HL Paper 2

Find the Cartesian equation of plane *Π* containing the points $\text{A}\,(6,\ 2,\ 1)$ and $\text{B}\,(3,\ -1,\ 1)$ and perpendicular to the plane $x + 2y - z - 6 = 0$.

Markscheme

METHOD 1

$\overrightarrow{\text{AB}} = \begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix}$ ***(A1)***

$\begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ ***M1A1***

$= \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix}$ ***A1***

$x - y - z = k$ ***M1***

$k = 3$ equation of plane *Π* is $x - y - z = 3$ or equivalent ***A1***

METHOD 2

let plane *Π* be $ax + by + cz = d$

attempt to form one or more simultaneous equations: ***M1***

$a + 2b - c = 0$ (1) ***A1***

$6a + 2b + c = d$ (2)

$3a - b + c = d$ (3) ***A1***

Note: Award second ***A1*** for equations (2) and (3).

attempt to solve ***M1***

EITHER

using GDC gives $a = \frac{d}{3},\ b = -\frac{d}{3},\ c = -\frac{d}{3}$ ***(A1)***

equation of plane *Π* is $x - y - z = 3$ or equivalent ***A1***

OR

row reduction ***M1***

equation of plane *Π* is $x - y - z = 3$ or equivalent ***A1***

[6 marks]

Examiners report

[N/A]

Given that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} \neq \mathbf{0}$ prove that $\boldsymbol{a} + \boldsymbol{c} = s\boldsymbol{b}$ where s is a scalar.

Markscheme

METHOD 1

$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c}$

$(\boldsymbol{a} \times \boldsymbol{b}) - (\boldsymbol{b} \times \boldsymbol{c}) = 0$

$(\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{c} \times \boldsymbol{b}) = 0$ ***M1A1***

$(\boldsymbol{a} + \boldsymbol{c}) \times \boldsymbol{b} = 0$ ***A1***

$(\boldsymbol{a} + \boldsymbol{c})$ is parallel to $\boldsymbol{b} \Rightarrow \boldsymbol{a} + \boldsymbol{c} = s\boldsymbol{b}$ ***R1AG***

Note: Condone absence of arrows, underlining, or other otherwise "correct" vector notation throughout this question.

Note: Allow "is in the same direction to", for the final ***R*** mark.

METHOD 2

$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} \Rightarrow \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} = \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix}$ ***M1A1***

$a_2b_3 - a_3b_2 = b_2c_3 - b_3c_2 \Rightarrow b_3(a_2 + c_2) = b_2(a_3 + c_3)$

$a_3b_1 - a_1b_3 = b_3c_1 - b_1c_3 \Rightarrow b_1(a_3 + c_3) = b_3(a_1 + c_1)$

$a_1b_2 - a_2b_1 = b_1c_2 - b_2c_1 \Rightarrow b_2(a_1 + c_1) = b_1(a_2 + c_2)$

$\frac{(a_1+c_1)}{b_1} = \frac{(a_2+c_2)}{b_2} = \frac{(a_3+c_3)}{b_3} = s$ ***A1***

$\Rightarrow a_1 + c_1 = sb_1$

$\Rightarrow a_2 + c_2 = sb_2$

$\Rightarrow a_3 + c_3 = sb_3$

$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = s\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ ***A1***

$\Rightarrow \boldsymbol{a} + \boldsymbol{c} = s\boldsymbol{b}$ ***AG***

[4 marks]

Examiners report

[N/A]

Consider the points P(−3, −1, 2) and Q(5, 5, 6).

a. Find a vector equation for the line, L_1, which passes through the points P and Q. [3]

The line L_2 has equation

$$r = \begin{pmatrix} -4 \\ 0 \\ 4 \end{pmatrix} + s\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}.$$

b. Show that L_1 and L_2 intersect at the point R(1, 2, 4). [4]

c. Find the acute angle between L_1 and L_2. [3]

d. Let S be a point on L_2 such that $\left|\overrightarrow{\mathrm{RP}}\right| = \left|\overrightarrow{\mathrm{RS}}\right|$. [6]

Show that one of the possible positions for S is $\mathrm{S}_1(-4, 0, 4)$ and find the coordinates of the other possible position, S_2.

e. Let S be a point on L_2 such that $\left|\overrightarrow{\mathrm{RP}}\right| = \left|\overrightarrow{\mathrm{RS}}\right|$. [4]

Find a vector equation of the line which passes through R and bisects $\mathrm{P\hat{R}S_1}$.

Markscheme

a. $\overrightarrow{\mathrm{PQ}} = \begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}$ ***(A1)***

equation of line: $r = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}$ (or equivalent) ***M1A1***

Note: Award ***M1A0*** if $\boldsymbol{r} =$ is omitted.

[3 marks]

b. **METHOD 1**

$x: \ -4 + 5s = -3 + 8t$

$y: \ 2s = -1 + 6t$

$z: \ 4 = 2 + 4t$ ***M1***

solving any two simultaneously ***M1***

$t = 0.5$, $s = 1$ (or equivalent) ***A1***

verification that these values give R when substituted into **both** equations (or that the three equations are consistent and that one gives R) ***R1***

METHOD 2

(1, 2, 4) is given by $t = 0.5$ for L_1 and $s = 1$ for L_2 ***M1A1A1***

because (1, 2, 4) is on both lines it is the point of intersection of the two lines ***R1***

[4 marks]

c. $\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = 26 = \sqrt{29} \times \sqrt{29}\cos\theta$ ***M1***

$\cos\theta = \frac{26}{29}$ ***(A1)***

$\theta = 0.459$ or $26.3°$ ***A1***

[3 marks]

d. $\overrightarrow{\mathrm{RP}} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ -2 \end{pmatrix}, \left|\overrightarrow{\mathrm{RP}}\right| = \sqrt{29}$ ***(M1)A1***

Note: This could also be obtained from $\left|0.5\begin{pmatrix}8\\6\\4\end{pmatrix}\right|$

EITHER

$\overrightarrow{\mathrm{RS}_1}=\begin{pmatrix}-4\\0\\4\end{pmatrix}-\begin{pmatrix}1\\2\\4\end{pmatrix}=\begin{pmatrix}-5\\-2\\0\end{pmatrix}, \left|\overrightarrow{\mathrm{RS}_1}\right|=\sqrt{29}$ ***A1***

$\therefore \overrightarrow{\mathrm{OS}_2}=\overrightarrow{\mathrm{OS}_1}+2\overrightarrow{\mathrm{S}_1\mathrm{R}}=\begin{pmatrix}-4\\0\\4\end{pmatrix}+2\begin{pmatrix}5\\2\\0\end{pmatrix}$ ***M1A1***

$\left(\text{or } \overrightarrow{\mathrm{OS}_2}=\overrightarrow{\mathrm{OR}}+\overrightarrow{\mathrm{S}_1\mathrm{R}}=\begin{pmatrix}1\\2\\4\end{pmatrix}+\begin{pmatrix}5\\2\\0\end{pmatrix}\right)$

$=\begin{pmatrix}6\\4\\4\end{pmatrix}$

S_2 is (6, 4, 4) ***A1***

OR

$\begin{pmatrix}-4+5s\\2s\\4\end{pmatrix}-\begin{pmatrix}1\\2\\4\end{pmatrix}=\begin{pmatrix}5s-5\\2s-2\\0\end{pmatrix}$ ***M1***

$(5s-5)^2+(2s-2)^2=29$ ***M1A1***

$29s^2-58s+29=29$

$s(s-2)=0,\ s=0,\ 2$

$(6,\ 4,\ 4)\ (\text{and }(-4,\ 0,\ 4))$ ***A1***

Note: There are several geometrical arguments possible using information obtained in previous parts, depending on what forms the previous answers had been given.

[6 marks]

e. **EITHER**

midpoint of $[\mathrm{PS}_1]$ is M(–3.5, –0.5, 3) ***M1A1***

$\overrightarrow{\mathrm{RM}}=\begin{pmatrix}-4.5\\-2.5\\-1\end{pmatrix}$ ***A1***

OR

$\overrightarrow{\mathrm{RS}_1}=\begin{pmatrix}-5\\-2\\0\end{pmatrix}$ ***M1***

the direction of the line is $\overrightarrow{\mathrm{RS}_1}+\overrightarrow{\mathrm{RP}}$

$\begin{pmatrix}-5\\-2\\0\end{pmatrix}+\begin{pmatrix}-4\\-3\\-2\end{pmatrix}=\begin{pmatrix}-9\\-5\\-2\end{pmatrix}$ ***M1A1***

THEN

the equation of the line is:

$r=\begin{pmatrix}1\\2\\4\end{pmatrix}+t\begin{pmatrix}9\\5\\2\end{pmatrix}$ or equivalent ***A1***

Note: Marks cannot be awarded for methods involving halving the angle, unless it is clear that the candidate considers also the equation of the plane of L_1 and L_2 to reduce the number of parameters involved to one (to obtain the vector equation of the required line).

[4 marks]

Examiners report

a. There were many good answers to part (a) showing a clear understanding of finding the vector equation of a line. Unfortunately this understanding was marred by many students failing to write the equation properly resulting in just 2 marks out of the 3. The most common response was of the form $L_1 = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$ which seemed a waste of a mark.

b. In part (b) many students failed to verify that the lines do indeed intersect.

c. Part (c) was very well done.

d. In part (d) most candidates were able to obtain the first three marks, but few were able to find the second point.

e. There were few correct answers to part (e).

The points P(−1, 2, − 3), Q(−2, 1, 0), R(0, 5, 1) and S form a parallelogram, where S is diagonally opposite Q.

a. Find the coordinates of S. [2]

b. The vector product $\overrightarrow{\text{PQ}} \times \overrightarrow{\text{PS}} = \begin{pmatrix} -13 \\ 7 \\ m \end{pmatrix}$. Find the value of m . [2]

c. Hence calculate the area of parallelogram PQRS. [2]

d. Find the Cartesian equation of the plane, $\prod_1$, containing the parallelogram PQRS. [3]

e. Write down the vector equation of the line through the origin (0, 0, 0) that is perpendicular to the plane $\prod_1$. [1]

f. Hence find the point on the plane that is closest to the origin. [3]

g. A second plane, $\prod_2$, has equation $x - 2y + z = 3$. Calculate the angle between the two planes. [4]

Markscheme

a. $\overrightarrow{\text{PQ}} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$, $\overrightarrow{\text{SR}} = \begin{pmatrix} 0 - x \\ 5 - y \\ 1 - z \end{pmatrix}$ ***(M1)***

point S = (1, 6, −2) ***A1***

[2 marks]

b. $\overrightarrow{\text{PQ}} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} \overrightarrow{\text{PS}} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$ ***A1***

$\overrightarrow{\text{PQ}} \times \overrightarrow{\text{PS}} = \begin{pmatrix} -13 \\ 7 \\ -2 \end{pmatrix}$

$m = -2$ ***A1***

[2 marks]

c. area of parallelogram PQRS $= \left|\overrightarrow{\mathrm{PQ}} \times \overrightarrow{\mathrm{PS}}\right| = \sqrt{(-13)^2 + 7^2 + (-2)^2}$ ***M1***

$= \sqrt{222} = 14.9$ ***A1***

[2 marks]

d. equation of plane is $-13x + 7y - 2z = d$ ***M1A1***

substituting any of the points given gives $d = 33$

$-13x + 7y - 2z = 33$ ***A1***

[3 marks]

e. equation of line is $\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -13 \\ 7 \\ -2 \end{pmatrix}$ ***A1***

Note: To get the ***A1*** must have $\boldsymbol{r} =$ or equivalent.

[1 mark]

f. $169\lambda + 49\lambda + 4\lambda = 33$ ***M1***

$\lambda = \frac{33}{222}\ (= 0.149\ldots)$ ***A1***

closest point is $\left(-\frac{143}{74}, \frac{77}{74}, -\frac{11}{37}\right)\ (= (-1.93,\ 1.04,\ \text{-}\ 0.297))$ ***A1***

[3 marks]

g. angle between planes is the same as the angle between the normals ***(R1)***

$\cos\theta = \frac{-13\times1+7\times-2-2\times1}{\sqrt{222}\times\sqrt{6}}$ ***M1A1***

$\theta = 143^\circ$ (accept $\theta = 37.4^\circ$ or 2.49 radians or 0.652 radians) ***A1***

[4 marks]

Examiners report

a. This was a multi-part question that was well answered by many candidates. Wrong answers to part (a) were mainly the result of failing to draw a diagram. Follow through benefitted many candidates. A high proportion of candidates lost the mark in (e) by not writing their answer as an equation in the form *r* = ...

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The vector equation of line l is given as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$.

Find the Cartesian equation of the plane containing the line l and the point A(4, − 2, 5) .

Markscheme

EITHER

l goes through the point (1, 3, 6) , and the plane contains A(4, –2, 5)

the vector containing these two points is on the plane, *i.e.*

$\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$ ***(M1)A1***

$\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 2 & -1 \\ -3 & 5 & 1 \end{vmatrix} = 7\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$ ***M1A1***

$\begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix} = 25$ ***(M1)***

hence, Cartesian equation of the plane is $7x + 4y + z = 25$ ***A1***

OR

finding a third point ***M1***

e.g. (0, 5, 5) ***A1***

three points are (1, 3, 6), (4, –2, 5), (0, 5, 5)

equation is $ax + by + cz = 1$

system of equations ***M1***

$a+3b+6c=1$
$4a-2b+5c=1$
$5b+5c=1$

$a=\frac{7}{25}$, $b=\frac{4}{25}$, $c=\frac{1}{25}$, from GDC ***M1A1***

so $\frac{7}{25}x+\frac{4}{25}y+\frac{1}{25}z=1$ ***A1***

or $7x+4y+z=25$

[6 marks]

Examiners report

There were many successful answers to this question, as would be expected. There seemed to be some students, however, that had not been taught the vector geometry section

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Consider the vectors $\boldsymbol{a}=\sin(2\alpha)\boldsymbol{i}-\cos(2\alpha)\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{b}=\cos\alpha\boldsymbol{i}-\sin\alpha\boldsymbol{j}-\boldsymbol{k}$, where $0<\alpha<2\pi$.

Let θ be the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

(a) Express $\cos\theta$ in terms of α.

(b) Find the acute angle α for which the two vectors are perpendicular.

(c) For $\alpha=\frac{7\pi}{6}$, determine the vector product of $\boldsymbol{a}$ and $\boldsymbol{b}$ and comment on the geometrical significance of this result.

Markscheme

(a) $\cos\theta=\frac{\boldsymbol{ab}}{|\boldsymbol{a}||\boldsymbol{b}|}=\frac{\sin 2\alpha\cos\alpha+\sin\alpha\cos 2\alpha-1}{\sqrt{2}\times\sqrt{2}}\left(=\frac{\sin 3\alpha-1}{2}\right)$ ***M1A1***

(b) $\boldsymbol{a}\perp\boldsymbol{b}\Rightarrow\cos\theta=0$ ***M1***

$\sin 2\alpha\cos\alpha+\sin\alpha\cos 2\alpha-1=0$

$\alpha=0.524\ \left(=\frac{\pi}{6}\right)$ ***A1***

(c)

METHOD 1

$$\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \sin 2\alpha & -\cos 2\alpha & 1\\ \cos\alpha & -\sin\alpha & -1\end{vmatrix}$$ ***(M1)***

assuming $\alpha=\frac{7\pi}{6}$

Note: Allow substitution at any stage.

$$\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 1\\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & -1\end{vmatrix}$$ ***A1***

$= \boldsymbol{i}\left(\frac{1}{2} - \frac{1}{2}\right) - \boldsymbol{j}\left(-\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) + \boldsymbol{k}\left(\frac{\sqrt{3}}{2} \times \frac{1}{2} - \frac{1}{2} \times \frac{\sqrt{3}}{2}\right)$

$= 0$ ***A1***

a and ***b*** are parallel ***R1***

Note: Accept decimal equivalents.

METHOD 2

from (a) $\cos\theta = -1$ (and $\sin\theta = 0$) ***M1A1***

$\boldsymbol{a} \times \boldsymbol{b} = 0$ ***A1***

a and ***b*** are parallel ***R1***

[8 marks]

Examiners report

This question was attempted by most candidates who in general were able to find the dot product of the vectors in part (a). However the simplification of the expression caused difficulties which affected the performance in part (b). Many candidates had difficulties in interpreting the meaning of $\boldsymbol{a} \times \boldsymbol{b} = 0$ in part (c).

The lines l_1 and l_2 are defined as

$l_1 : \frac{x-1}{3} = \frac{y-5}{2} = \frac{z-12}{-2}$

$l_2 : \frac{x-1}{8} = \frac{y-5}{11} = \frac{z-12}{6}.$

The plane π contains both l_1 and l_2.

a. Find the Cartesian equation of π. [4]

b. The line l_3 passing through the point $(4,\ 0,\ 8)$ is perpendicular to π. [4]

Find the coordinates of the point where l_3 meets π.

Markscheme

a. attempting to find a normal to π *eg* $\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 8 \\ 11 \\ 6 \end{pmatrix}$ ***(M1)***

$\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 8 \\ 11 \\ 6 \end{pmatrix} = 17\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ ***(A1)***

$r \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 12 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ ***M1***

$2x - 2y + z = 4$ (or equivalent) ***A1***

[4 marks]

b. $l_3 : r = \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix} + t \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$ ***(A1)***

attempting to solve $\begin{pmatrix} 4+2t \\ -2t \\ 8+t \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 4$ for t *ie* $9t + 16 = 4$ for t ***M1***

$t = -\frac{4}{3}$ ***A1***

$\left(\frac{4}{3}, \frac{8}{3}, \frac{20}{3}\right)$ ***A1***

[4 marks]

Total [8 marks]

Examiners report

a. Part (a) was reasonably well done. Some candidates made numerical errors when attempting to find a normal to π.

b. In part (b), a number of candidates were awarded follow through marks from numerical errors committed in part (a).

(a) If $a = 4$ find the coordinates of the point of intersection of the three planes.

(b) (i) Find the value of a for which the planes do not meet at a unique point.

(ii) For this value of a show that the three planes do not have any common point.

Markscheme

(a) let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 5 & -1 & 4 \end{pmatrix}$, $\boldsymbol{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and $\boldsymbol{B} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$ ***(M1)***

point of intersection is $\left(\frac{11}{12}, \frac{7}{12}, \frac{1}{4}\right)$ or (or $(0.917, 0.583, 0.25)$) ***A1***

(b) **METHOD 1**

(i) $\det \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 5 & -1 & 4 \end{pmatrix} = 0$ ***M1***

$-3a + 24 = 0$ ***(A1)***

$a = 8$ ***A1 N1***

(ii) consider the augmented matrix $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & 4 & 5 \end{array}\right)$ ***M1***

use row reduction to obtain $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & 0 & -1 \end{array}\right)$ or $\left(\begin{array}{ccc|c} 1 & 0 & \frac{5}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ (or equivalent) ***A1*** any valid reason ***R1***

(*e.g.* as the last row is not all zeros, the planes do not meet) ***N0***

METHOD 2

use of row reduction (or equivalent manipulation of equations) ***M1***

e.g. $\begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & a & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & -6 & a-10 & -5 \end{pmatrix}$ ***A1A1***

Note: Award an ***A1*** for each correctly reduced row.

(i) $a - 10 = -2 \Rightarrow a = 8$ ***M1A1*** ***N1***

(ii) when $a = 8$, row $3 \neq 2 \times$ row 2 ***R1*** ***N0***

[8 marks]

Examiners report

Few students were able to do this question efficiently. Many students were able to do part (a) by manipulating equations, whereas calculator methods would yield the solution quickly and easily. Part (b) was poorly attempted and it was apparent that many students used a lot of time manipulating equations without real understanding of what they were looking for.

The points A and B have position vectors $\overrightarrow{\mathrm{OA}} = \begin{Bmatrix} 1 \\ 2 \\ -2 \end{Bmatrix}$ and $\overrightarrow{\mathrm{OB}} = \begin{Bmatrix} 1 \\ 0 \\ 2 \end{Bmatrix}$.

a. Find $\overrightarrow{\mathrm{OA}} \times \overrightarrow{\mathrm{OB}}$. [2]

b. Hence find the area of the triangle OAB. [2]

Markscheme

a. $\overrightarrow{\mathrm{OA}} \times \overrightarrow{\mathrm{OB}} = \begin{pmatrix} 4 \\ -4 \\ -2 \end{pmatrix}$ ***(M1)A1***

Note: ***M1A0*** can be awarded for attempt at a correct method **shown**, or correct method implied by the digits 4, 4, 2 found in the correct order.

[2 marks]

b. $\text{area} = \frac{1}{2}\sqrt{4^2 + 4^2 + 2^2} = 3$ ***M1A1***

[2 marks]

Examiners report

a. Generally well done. Most students were able to obtain full marks on this question. Most of the errors made were due to careless mistakes.

b. Generally well done. Most students were able to obtain full marks on this question. Most of the errors made were due to careless mistakes. A few students did not take notice of the "hence" in part (b) and were consequently not able to obtain the marks.

The points A and B have coordinates (1, 2, 3) and (3, 1, 2) relative to an origin O.

a. (i) Find $\overrightarrow{\text{OA}} \times \overrightarrow{\text{OB}}$. [5]

(ii) Determine the area of the triangle OAB.

(iii) Find the Cartesian equation of the plane OAB.

b. (i) Find the vector equation of the line L_1 containing the points A and B. [7]

(ii) The line L_2 has vector equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$.

Determine whether or not L_1 and L_2 are skew.

Markscheme

a. (i) $\overrightarrow{\text{OA}} \times \overrightarrow{\text{OB}} = \boldsymbol{i} + 7\boldsymbol{j} - 5\boldsymbol{k}$ ***A1***

(ii) area $= \frac{1}{2}|\boldsymbol{i} + 7\boldsymbol{j} - 5\boldsymbol{k}| = \frac{5\sqrt{3}}{2}(4.33)$ ***M1A1***

(iii) equation of plane is $x + 7y - 5z = k$ ***M1***

$x + 7y - 5z = 0$ ***A1***

[5 marks]

b. (i) direction of line = $(3\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) - (\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) = 2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ ***M1A1***

equation of line is

$\boldsymbol{r} = (\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) + \lambda(2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$ ***A1***

(ii) at a point of intersection,

$1 + 2\lambda = 2 + \mu$

$2 - \lambda = 4 + 3\mu$ ***M1A1***

$3 - \lambda = 3 + 2\mu$

solving the 2$^{\text{nd}}$ and 3$^{\text{rd}}$ equations, $\lambda = 4$, $\mu = -2$ ***A1***

these values do not satisfy the 1$^{\text{st}}$ equation so the lines are skew ***R1***

[7 marks]

Examiners report

a. [N/A]
b. [N/A]

Consider the planes $\pi_1 : x - 2y - 3z = 2$ and $\pi_2 : 2x - y - z = k$.

a. Find the angle between the planes $\pi_1$and π_2 . [4]

b. The planes π_1 and π_2 intersect in the line L_1 . Show that the vector equation of L_1 is $r = \begin{pmatrix} 0 \\ 2-3k \\ 2k-2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$ [5]

c. The line L_2 has Cartesian equation $5 - x = y + 3 = 2 - 2z$. The lines L_1 and L_2 intersect at a point X. Find the coordinates of X. [5]

d. Determine a Cartesian equation of the plane π_3 containing both lines L_1 and L_2 . [5]

e. Let Y be a point on L_1 and Z be a point on L_2 such that XY is perpendicular to YZ and the area of the triangle XYZ is 3. Find the perimeter of the triangle XYZ. [5]

Markscheme

a. **Note:** Accept alternative notation for vectors (*eg* $\langle a,b,c\rangle$ or (a,b,c)).

$$\boldsymbol{n} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \text{ and } \boldsymbol{m} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \quad \textbf{\textit{(A1)}}$$

$\cos\theta = \frac{\boldsymbol{n}\cdot\boldsymbol{m}}{|\boldsymbol{n}||\boldsymbol{m}|}$ ***(M1)***

$\cos\theta = \frac{2+2+3}{\sqrt{1+4+9}\sqrt{4+1+1}} = \frac{7}{\sqrt{14}\sqrt{6}}$ ***A1***

$\theta = 40.2^\circ \quad (0.702 \text{ rad})$ ***A1***

[4 marks]

b. **Note:** Accept alternative notation for vectors (*eg* $\langle a,b,c\rangle$ or (a,b,c)).

METHOD 1

eliminate z from $x - 2y - 3z = 2$ and $2x - y - z = k$

$5x - y = 3k - 2 \Rightarrow x = \frac{y-(2-3k)}{5}$ ***M1A1***

eliminate y from $x - 2y - 3z = 2$ and $2x - y - z = k$

$3x + z = 2k - 2 \Rightarrow x = \frac{z-(2k-2)}{-3}$ ***A1***

$x = t$, $y = (2 - 3k) + 5t$ and $z = (2k - 2) - 3t$ ***A1A1***

$$r = \begin{pmatrix} 0 \\ 2-3k \\ 2k-2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix} \quad \textbf{\textit{AG}}$$

[5 marks]

METHOD 2

$$\begin{pmatrix}1\\-2\\-3\end{pmatrix}\times\begin{pmatrix}2\\-1\\-1\end{pmatrix}=\begin{pmatrix}-1\\-5\\3\end{pmatrix}\Rightarrow \text{direction is }\begin{pmatrix}1\\5\\-3\end{pmatrix}$$ ***M1A1***

Let $x = 0$

$0 - 2y - 3z = 2$ and $2 \times 0 - y - z = k$ ***(M1)***

solve simultaneously ***(M1)***

$y = 2 - 3k$ and $z = 2k - 2$ ***A1***

therefore $\mathbf{r} = \begin{pmatrix}0\\2-3k\\2k-2\end{pmatrix}+t\begin{pmatrix}1\\5\\-3\end{pmatrix}$ ***AG***

[5 marks]

METHOD 3

substitute $x = t,\ y = (2 - 3k) + 5t$ and $z = (2k - 2) - 3t$ into π_1 and π_2 ***M1***

for $\pi_1 : t - 2(2 - 3k + 5t) - 3(2k - 2 - 3t) = 2$ ***A1***

for $\pi_2 : 2t - (2 - 3k + 5t) - (2k - 2 - 3t) = k$ ***A1***

the planes have a unique line of intersection ***R2***

therefore the line is $r = \begin{pmatrix}0\\2-3k\\2k-2\end{pmatrix}+t\begin{pmatrix}1\\5\\-3\end{pmatrix}$ ***AG***

[5 marks]

c. **Note:** Accept alternative notation for vectors (*eg* $\langle a, b, c\rangle$ or (a, b, c)).

$5 - t = (2 - 3k + 5t) + 3 = 2 - 2(2k - 2 - 3t)$ ***M1A1***
Note: Award ***M1A1*** if candidates use vector or parametric equations of L_2
eg $\begin{pmatrix}0\\2-3k\\2k-2\end{pmatrix}+t\begin{pmatrix}1\\5\\-3\end{pmatrix}=\begin{pmatrix}5\\-3\\1\end{pmatrix}+s\begin{pmatrix}-2\\2\\-1\end{pmatrix}$ or $\Rightarrow \begin{cases}t=5-2s\\2-3k+5t=-3+2s\\2k-2-3t=1+s\end{cases}$

solve simultaneously ***M1***
$k = 2,\ t = 1\ (s = 2)$ ***A1***
intersection point $(1, 1, -1)$ ***A1***

[5 marks]

d. **Note:** Accept alternative notation for vectors (*eg* $\langle a, b, c\rangle$ or (a, b, c)).

$\vec{l}_2 = \begin{pmatrix}2\\-2\\1\end{pmatrix}$ ***A1***

$\vec{l}_1\times\vec{l}_2 = \begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\1&5&-3\\2&-2&1\end{vmatrix}=\begin{pmatrix}-1\\-7\\-12\end{pmatrix}$ ***(M1)A1***

$\boldsymbol{r}\cdot\begin{pmatrix}1\\7\\12\end{pmatrix}=\begin{pmatrix}1\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\7\\12\end{pmatrix}$ ***(M1)***
$x + 7y + 12z = -4$ ***A1***
[5 marks]

e. **Note:** Accept alternative notation for vectors (*eg* $\langle a, b, c\rangle$ or (a, b, c)).

Let θ be the angle between the lines $\vec{l}_1 = \begin{pmatrix}1\\5\\-3\end{pmatrix}$ and $\vec{l}_2 = \begin{pmatrix}2\\-2\\1\end{pmatrix}$

$\cos\theta = \frac{|2-10-3|}{\sqrt{35}\sqrt{9}} \Rightarrow \theta = 0.902334 \ldots 51.699\ldots^\circ)$ ***(M1)***

as the triangle XYZ has a right angle at Y,

$\text{XZ} = a \Rightarrow \text{YZ} = a\sin\theta$ and $\text{XY} = a\cos\theta$ ***(M1)***

$\text{area} = 3 \Rightarrow \frac{a^2\sin\theta\cos\theta}{2} = 3$ ***(M1)***

$a = 3.5122\ldots$ ***(A1)***

$\text{perimeter} = a + a\sin\theta + a\cos\theta = 8.44537\ldots = 8.45$ ***A1***

Note: If candidates attempt to find coordinates of Y and Z award ***M1*** for expression of vector YZ in terms of two parameters, ***M1*** for attempt to use perpendicular condition to determine relation between parameters, ***M1*** for attempt to use the area to find the parameters and ***A2*** for final answer.

[5 marks]

Examiners report

a. Although this was the last question in part B, it was answered surprisingly well by many candidates, except for part (e). Even those who had not done so well elsewhere often gained a number of marks in some parts of the question. Nevertheless the presence of parameters seemed to have blocked the abilities of weaker candidates to solve situations in which vectors were involved. Mathematical skills for this particular question were sometimes remarkable, however, calculations proved incomplete due to the way that planes were presented. Most candidates found a correct angle in part (a). Occasional arithmetic errors in calculating the magnitude of a vector and dot product occurred. In part (b) the vector product approach was popular. In some case candidates simply verified the result by substitution. There was a lot of simultaneous equation solving, much of which was not very pretty. In part (c), a number of candidates made errors when attempting to solve a system of equations involving parameters. Many of the results for the point were found in terms of *k*. It was notorious that candidates did not use their GDC to try to find the coordinates of the intersection point between lines. In part (d), a number of candidates used an incorrect point but this part was often done well.

Very few excellent answers to part (e) were seen using an efficient method. Most candidates attempted methods involving heavy algebraic manipulation and had little success in this part of the question.

b. Although this was the last question in part B, it was answered surprisingly well by many candidates, except for part (e). Even those who had not done so well elsewhere often gained a number of marks in some parts of the question. Nevertheless the presence of parameters seemed to have blocked the abilities of weaker candidates to solve situations in which vectors were involved. Mathematical skills for this particular question were sometimes remarkable, however, calculations proved incomplete due to the way that planes were presented. Most candidates found a correct angle in part (a). Occasional arithmetic errors in calculating the magnitude of a vector and dot product occurred. In part (b) the vector product approach was popular. In some case candidates simply verified the result by substitution. There was a lot of simultaneous equation solving, much of which was not very pretty. In part (c), a number of candidates made errors when attempting to solve a system of equations involving parameters. Many of the results for the point were found in terms of *k*. It was notorious that candidates did not use their GDC to try to find the coordinates of the intersection point between lines. In part (d), a number of candidates used an incorrect point but this part was often done well.

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Very few excellent answers to part (e) were seen using an efficient method. Most candidates attempted methods involving heavy algebraic manipulation and had little success in this part of the question.

Find the vector equation of the line of intersection of the three planes represented by the following system of equations.

$$2x - 7y + 5z = 1$$

$$6x + 3y - z = -1$$

$$-14x - 23y + 13z = 5$$

Markscheme

METHOD 1

(from GDC)

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{6} & -\frac{1}{12} \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{6} \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \textit{(M1)}$$

$x + \frac{1}{6}\lambda = -\frac{1}{12}$ ***A1***

$y - \frac{2}{3}\lambda = -\frac{1}{6}$ ***A1***

$\boldsymbol{r} = \left(-\frac{1}{12}\boldsymbol{i} - \frac{1}{6}\boldsymbol{j}\right) + \lambda\left(-\frac{1}{6}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} + \boldsymbol{k}\right)$ ***A1A1A1*** ***N3***

[6 marks]

METHOD 2

(Elimination method either for equations or row reduction of matrix)

Eliminating one of the variables ***M1A1***

Finding a point on the line ***(M1)A1***

Finding the direction of the line ***M1***

The **vector** equation of the line ***A1*** ***N3***

[6 marks]

Examiners report

A large number of candidates did not use their GDC in this question. Some candidates who attempted analytical solutions looked for a point solution although the question specifically states that the planes intersect in a line. Other candidates eliminated one variable and then had no clear strategy for proceeding with the solution.

Some candidates failed to write '*r* =', and others did not give the equation in vector form.

Two submarines A and B have their routes planned so that their positions at time *t* hours, $0 \le t < 20$, would be defined by the position vectors $\boldsymbol{r}_A = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ -0.15 \end{pmatrix}$ and $\boldsymbol{r}_B = \begin{pmatrix} 0 \\ 3.2 \\ -2 \end{pmatrix} + t\begin{pmatrix} -0.5 \\ 1.2 \\ 0.1 \end{pmatrix}$ relative to a fixed point on the surface of the ocean (all lengths are in kilometres).

To avoid the collision submarine B adjusts its velocity so that its position vector is now given by

$$\boldsymbol{r}_B = \begin{pmatrix} 0 \\ 3.2 \\ -2 \end{pmatrix} + t\begin{pmatrix} -0.45 \\ 1.08 \\ 0.09 \end{pmatrix}.$$

a. Show that the two submarines would collide at a point P and write down the coordinates of P. [4]

b.i. Show that submarine B travels in the same direction as originally planned. [1]

b.ii. Find the value of t when submarine B passes through P. [2]

c.i. Find an expression for the distance between the two submarines in terms of t. [5]

c.ii. Find the value of t when the two submarines are closest together. [2]

c.iii. Find the distance between the two submarines at this time. [1]

Markscheme

a. $\boldsymbol{r}_A = \boldsymbol{r}_B$ ***(M1)***

$2 - t = -0.5t \Rightarrow t = 4$ **A1**

checking $t = 4$ satisfies $4 + t = 3.2 + 1.2t$ and $-1 - 0.15t = -2 + 0.1t$ ***R1***

P(−2, 8, −1.6) ***A1***

Note: Do not award final ***A1*** if answer given as column vector.

[4 marks]

b.i. $0.9 \times \begin{pmatrix} -0.5 \\ 1.2 \\ 0.1 \end{pmatrix} = \begin{pmatrix} -0.45 \\ 1.08 \\ 0.09 \end{pmatrix}$ ***A1***

Note: Accept use of cross product equalling zero.

hence in the same direction ***AG***

[1 mark]

b.ii. $\begin{pmatrix} -0.45t \\ 3.2 + 1.08t \\ -2 + 0.09t \end{pmatrix} = \begin{pmatrix} -2 \\ 8 \\ -1.6 \end{pmatrix}$ ***M1***

Note: The ***M1*** can be awarded for any one of the resultant equations.

$\Rightarrow t = \frac{40}{9} = 4.44\ldots$ ***A1***

[2 marks]

c.i. $\boldsymbol{r}_A - \boldsymbol{r}_B = \begin{pmatrix} 2 - t \\ 4 + t \\ -1 - 0.15t \end{pmatrix} - \begin{pmatrix} -0.45t \\ 3.2 + 1.08t \\ -2 + 0.09t \end{pmatrix}$ ***(M1)(A1)***

$= \begin{pmatrix} 2 - 0.55t \\ 0.8 - 0.08t \\ 1 - 0.24t \end{pmatrix}$ ***(A1)***

Note: Accept $\boldsymbol{r}_A - \boldsymbol{r}_B$.

distance $D = \sqrt{(2 - 0.55t)^2 + (0.8 - 0.08t)^2 + (1 - 0.24t)^2}$ ***M1A1***

$\left(= \sqrt{8.64 - 2.688t + 0.317t^2}\right)$

[5 marks]

c.ii. minimum when $\frac{\mathrm{d}D}{\mathrm{d}t} = 0$ ***(M1)***

$t = 3.83$ ***A1***

[2 marks]

c.iii. 0.511 (km) ***A1***

[1 mark]

Examiners report

a. [N/A]
b.i. [N/A]
b.ii. [N/A]
c.i. [N/A]
c.ii. [N/A]
c.iii. [N/A]

The points A, B and C have the following position vectors with respect to an origin O.

$$\overrightarrow{OA} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$$

$$\overrightarrow{OB} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$$

$$\overrightarrow{OC} = \boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}$$

The plane Π_2 contains the points O, A and B and the plane Π_3 contains the points O, A and C.

a. Find the vector equation of the line (BC). [3]

b. Determine whether or not the lines (OA) and (BC) intersect. [6]

c. Find the Cartesian equation of the plane Π_1, which passes through C and is perpendicular to $\overrightarrow{OA}$. [3]

d. Show that the line (BC) lies in the plane Π_1. [2]

e. Verify that 2$\boldsymbol{j}$ + $\boldsymbol{k}$ is perpendicular to the plane Π_2. [3]

f. Find a vector perpendicular to the plane Π_3. [1]

g. Find the acute angle between the planes Π_2 and Π_3. [4]

Markscheme

a. $\overrightarrow{BC} = (\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}) - (2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) = -\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$ ***(A1)***

$\boldsymbol{r} = (2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) + \lambda(-\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k})$

(or $\boldsymbol{r} = (\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}) + \lambda(-\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k})$) ***(M1)A1***

Note: Do not award ***A1*** unless $\boldsymbol{r}$ = or equivalent correct notation seen.

[3 marks]

b. attempt to write in parametric form using two different parameters **AND** equate ***M1***

$2\mu = 2 - \lambda$

$\mu = -1 + 4\lambda$

$-2\mu = 2 + \lambda$ ***A1***

attempt to solve first pair of simultaneous equations for two parameters ***M1***

solving first two equations gives $\lambda = \frac{4}{9},\ \mu = \frac{7}{9}$ ***(A1)***

substitution of these two values in third equation ***(M1)***

since the values do not fit, the lines do not intersect ***R1***

Note: Candidates may note that adding the first and third equations immediately leads to a contradiction and hence they can immediately deduce that the lines do not intersect.

[6 marks]

c. **METHOD 1**

plane is of the form $\boldsymbol{r} \bullet (2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = d$ ***(A1)***

$d = (\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}) \bullet (2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = -1$ ***(M1)***

hence Cartesian form of plane is $2x + y - 2z = -1$ ***A1***

METHOD 2

plane is of the form $2x + y - 2z = d$ ***(A1)***

substituting $(1,\ 3,\ 3)$ (to find gives $2 + 3 - 6 = -1$) ***(M1)***

hence Cartesian form of plane is $2x + y - 2z = -1$ ***A1***

[3 marks]

d. **METHOD 1**

attempt scalar product of direction vector BC with normal to plane ***M1***

$(-\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}) \bullet (2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = -2 + 4 - 2$

$= 0$ ***A1***

hence BC lies in Π_1 ***AG***

METHOD 2

substitute eqn of line into plane ***M1***

$$\text{line } r = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}.\ \text{Plane } \pi_1 : 2x + y - 2z = -1$$

$2(2 - \lambda) + (-1 + 4\lambda) - 2(2 + \lambda)$

$= -1$ ***A1***

hence BC lies in Π_1 ***AG***

Note: Candidates may also just substitute $2i - j + 2k$ into the plane since they are told C lies on π_1.

Note: Do not award ***A1FT***.

[2 marks]

e. **METHOD 1**

applying scalar product to $\overrightarrow{\mathrm{OA}}$ and $\overrightarrow{\mathrm{OB}}$ ***M1***

$(2\boldsymbol{j} + \boldsymbol{k}) \bullet (2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = 0$ ***A1***

$(2\boldsymbol{j} + \boldsymbol{k}) \bullet (2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) = 0$ ***A1***

METHOD 2

attempt to find cross product of $\overrightarrow{\mathrm{OA}}$ and $\overrightarrow{\mathrm{OB}}$ ***M1***

plane Π_2 has normal $\overrightarrow{\mathrm{OA}} \times \overrightarrow{\mathrm{OB}} = -8\boldsymbol{j} - 4\boldsymbol{k}$ ***A1***

since $-8\boldsymbol{j} - 4\boldsymbol{k} = -4(2\boldsymbol{j} + \boldsymbol{k})$, $2\boldsymbol{j} + \boldsymbol{k}$ is perpendicular to the plane Π_2 ***R1***

[3 marks]

f. plane Π_3 has normal $\overrightarrow{\mathrm{OA}} \times \overrightarrow{\mathrm{OC}} = 9\boldsymbol{i} - 8\boldsymbol{j} + 5\boldsymbol{k}$ ***A1***

[1 mark]

g. attempt to use dot product of normal vectors ***(M1)***

$\cos\theta = \frac{(2j+k)\bullet(9i-8j+5k)}{|2j+k||9i-8j+5k|}$ ***(M1)***

$= \frac{-11}{\sqrt{5}\sqrt{170}}\ \ (= -0.377\ldots)$ ***(A1)***

Note: Accept $\frac{11}{\sqrt{5}\sqrt{170}}$. acute angle between planes $= 67.8^\circ\ \ (= 1.18^\circ)$ ***A1***

[4 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]
e. [N/A]
f. [N/A]
g. [N/A]

OACB is a parallelogram with $\overrightarrow{\mathrm{OA}} = \boldsymbol{a}$ and $\overrightarrow{\mathrm{OB}} = \boldsymbol{b}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors.

a. Show that [4]

(i) $\left|\overrightarrow{\mathrm{OC}}\right|^2 = |\boldsymbol{a}|^2 + 2\boldsymbol{a} \bullet \boldsymbol{b} + |\boldsymbol{b}|^2$;

(ii) $\left|\overrightarrow{\mathrm{AB}}\right|^2 = |\boldsymbol{a}|^2 - 2\boldsymbol{a} \bullet \boldsymbol{b} + |\boldsymbol{b}|^2$.

b. Given that $\left|\overrightarrow{\mathrm{OC}}\right| = \left|\overrightarrow{\mathrm{AB}}\right|$, prove that OACB is a rectangle. [4]

Markscheme

a. **METHOD 1**

$$\left|\overrightarrow{\mathrm{OC}}\right|^2 = \overrightarrow{\mathrm{OC}} \bullet \overrightarrow{\mathrm{OC}}$$

$= (\boldsymbol{a} + \boldsymbol{b}) \bullet (\boldsymbol{a} + \boldsymbol{b})$ ***A1***

$= \boldsymbol{a} \bullet \boldsymbol{a} + \boldsymbol{a} \bullet \boldsymbol{b} + \boldsymbol{b} \bullet \boldsymbol{a} + \boldsymbol{b} \bullet \boldsymbol{b}$ ***A1***

$= |\boldsymbol{a}|^2 + 2\boldsymbol{a} \bullet \boldsymbol{b} + |\boldsymbol{b}|^2$ ***AG***

METHOD 2

$$\left|\overrightarrow{\mathrm{OC}}\right|^2 = \left|\overrightarrow{\mathrm{OA}}\right|^2 + \left|\overrightarrow{\mathrm{OB}}\right|^2 - 2\left|\overrightarrow{\mathrm{OA}}\right|\left|\overrightarrow{\mathrm{OB}}\right|\cos(\mathrm{O\hat{A}C})$$ ***A1***

$$\left|\overrightarrow{\mathrm{OA}}\right|\left|\overrightarrow{\mathrm{OB}}\right|\cos(\mathrm{O\hat{A}C}) = -(\boldsymbol{a} \bullet \boldsymbol{b})$$ ***A1***

$$\left|\overrightarrow{\mathrm{OC}}\right|^2 = |\boldsymbol{a}|^2 + 2\boldsymbol{a} \bullet \boldsymbol{b} + |\boldsymbol{b}|^2$$ ***AG***

(ii) **METHOD 1**

$$\left|\overrightarrow{\mathrm{AB}}\right|^2 = \overrightarrow{\mathrm{AB}} \bullet \overrightarrow{\mathrm{AB}}$$

$= (\boldsymbol{b} - \boldsymbol{a}) \bullet (\boldsymbol{b} - \boldsymbol{a})$ ***A1***

$= \boldsymbol{b} \bullet \boldsymbol{b} - \boldsymbol{b} \bullet \boldsymbol{a} - \boldsymbol{a} \bullet \boldsymbol{b} + \boldsymbol{a} \bullet \boldsymbol{a}$ ***A1***

$= |\boldsymbol{a}|^2 - 2\boldsymbol{a} \bullet \boldsymbol{b} + |\boldsymbol{b}|^2$ ***AG***

METHOD 2

$$\left|\overrightarrow{\mathrm{AB}}\right|^2 = \left|\overrightarrow{\mathrm{AC}}\right|^2 + \left|\overrightarrow{\mathrm{BC}}\right|^2 - 2\left|\overrightarrow{\mathrm{AC}}\right|\left|\overrightarrow{\mathrm{BC}}\right|\cos(\mathrm{A\hat{C}B})$$ ***A1***

$$\left|\overrightarrow{\mathrm{AC}}\right|\left|\overrightarrow{\mathrm{BC}}\right|\cos(\mathrm{A\hat{C}B}) = \boldsymbol{a} \bullet \boldsymbol{b}$$ ***A1***

$$\left|\overrightarrow{\mathrm{AB}}\right|^2 = |\boldsymbol{a}|^2 - 2\boldsymbol{a} \bullet \boldsymbol{b} + |\boldsymbol{b}|^2$$ ***AG***

[4 marks]

b. $\left|\overrightarrow{\mathrm{OC}}\right| = \left|\overrightarrow{\mathrm{AB}}\right| \Rightarrow \left|\overrightarrow{\mathrm{OC}}\right|^2 = \left|\overrightarrow{\mathrm{AB}}\right|^2 \Rightarrow |\boldsymbol{a}|^2 + 2\boldsymbol{a} \bullet \boldsymbol{b} + |\boldsymbol{b}|^2 = |\boldsymbol{a}|^2 - 2\boldsymbol{a} \bullet \boldsymbol{b} + |\boldsymbol{b}|^2$ ***R1(M1)***

Note: Award ***R1*** for $\left|\overrightarrow{\mathrm{OC}}\right| = \left|\overrightarrow{\mathrm{AB}}\right| \Rightarrow \left|\overrightarrow{\mathrm{OC}}\right|^2 = \left|\overrightarrow{\mathrm{AB}}\right|^2$ and ***(M1)*** for $|\boldsymbol{a}|^2 + 2\boldsymbol{a} \bullet \boldsymbol{b} + |\boldsymbol{b}|^2 = |\boldsymbol{a}|^2 - 2\boldsymbol{a} \bullet \boldsymbol{b} + |\boldsymbol{b}|^2$.

$\boldsymbol{a} \bullet \boldsymbol{b} = 0$ ***A1***

hence OACB is a rectangle (***a*** and ***b*** both non-zero)

with adjacent sides at right angles ***R1***

Note: Award ***R1(M1)A0R1*** if the dot product has not been used.

[4 marks]

Examiners report

a. In part (a), a significant number of candidates either did not use correct vector notation or simply did not use vector notation at all. A large number of candidates who appeared to adopt a scalar product approach, did not use the scalar product 'dot' and represented ***a* • *b*** as ***ab***. A few candidates successfully used the cosine rule with correct vector notation. A small number of candidates expressed ***a*** and ***b*** in general component form. In part (a) (ii), quite a number of candidates expressed $\overrightarrow{\mathrm{AB}}$ as ***a*** – ***b*** rather than as ***b*** – ***a***.

b. In part (b), some very well structured proofs were offered by a small number of candidates.

Ed walks in a straight line from point $\mathrm{P}(-1,\ 4)$ to point $\mathrm{Q}(4,\ 16)$ with constant speed.

Ed starts from point P at time $t = 0$ and arrives at point Q at time $t = 3$, where t is measured in hours.

Given that, at time t, Ed's position vector, relative to the origin, can be given in the form, $r = a + tb$,

a. find the vectors a and b. [3]

b. Roderick is at a point $\mathrm{C}(11,\ 9)$. During Ed's walk from P to Q Roderick wishes to signal to Ed. He decides to signal when Ed is at the closest point to C. [5]

Find the time when Roderick signals to Ed.

Markscheme

a. $a = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ ***A1***

$b = \frac{1}{3}\left(\begin{pmatrix} 4 \\ 16 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \end{pmatrix}\right) = \begin{pmatrix} \frac{5}{3} \\ 4 \end{pmatrix}$ ***(M1)A1***

[3 marks]

b. **METHOD 1**

Roderick must signal in a direction vector perpendicular to Ed's path. ***(M1)***

the equation of the signal is $\mathbf{s} = \begin{pmatrix} 11 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} -12 \\ 5 \end{pmatrix}$ (or equivalent) ***A1***

$\begin{pmatrix} -1 \\ 4 \end{pmatrix} + \frac{t}{3}\begin{pmatrix} 5 \\ 12 \end{pmatrix} = \begin{pmatrix} 11 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} -12 \\ 5 \end{pmatrix}$ ***M1***

$\frac{5}{3}t + 12\lambda = 12$ and $4t - 5\lambda = 5$ ***M1***

$t = 2.13 \quad \left(= \frac{360}{169}\right)$ ***A1***

METHOD 2

$\begin{pmatrix} 5 \\ 12 \end{pmatrix} \bullet \left(\begin{pmatrix} 11 \\ 9 \end{pmatrix} - \begin{pmatrix} -1 + \frac{5}{3}t \\ 4 + 4t \end{pmatrix}\right) = 0$ (or equivalent) ***M1A1A1***

Note: Award the ***M1*** for an attempt at a scalar product equated to zero, ***A1*** for the first factor and ***A1*** for the complete second factor.

attempting to solve for t ***(M1)***

$t = 2.13 \quad \left(\frac{360}{169}\right)$ ***A1***

METHOD 3

$x = \sqrt{\left(12 - \frac{5t}{3}\right)^2 + (5 - 4t)^2}$ (or equivalent) $\left(x^2 = \left(12 - \frac{5t}{3}\right)^2 + (5 - 4t)^2\right)$ ***M1A1A1***

Note: Award ***M1*** for use of Pythagoras' theorem, ***A1*** for $\left(12 - \frac{5t}{3}\right)^2$ and ***A1*** for $(5 - 4t)^2$.

attempting (graphically or analytically) to find t such that $\frac{dx}{dt} = 0 \left(\frac{d(x^2)}{dt} = 0\right)$ ***(M1)***

$t = 2.13 \quad \left(= \frac{360}{169}\right)$ ***A1***

METHOD 4

$\cos\theta = \frac{\begin{pmatrix}12\\5\end{pmatrix} \bullet \begin{pmatrix}5\\12\end{pmatrix}}{\left|\begin{pmatrix}12\\5\end{pmatrix}\right|\left|\begin{pmatrix}5\\12\end{pmatrix}\right|} = \frac{120}{169}$ ***M1A1***

Note: Award ***M1*** for attempting to calculate the scalar product.

$\frac{120}{13} = \frac{t}{3}\left|\begin{pmatrix}5\\12\end{pmatrix}\right|$ (or equivalent) ***(A1)***

attempting to solve for t ***(M1)***

$t = 2.13 \quad \left(= \frac{360}{169}\right)$ ***A1***

[5 marks]

Total [8 marks]

Examiners report

a. [N/A]
b. [N/A]

---

a. Find the values of k for which the following system of equations has no solutions and the value of k for the system to have an infinite number of solutions. [5]

$$x - 3y + z = 3$$

$$x + 5y - 2z = 1$$

$$16y - 6z = k$$

b. Given that the system of equations can be solved, find the solutions in the form of a vector equation of a line, $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$, where the components of $\boldsymbol{b}$ are integers. [7]

c. The plane ÷ is parallel to both the line in part (b) and the line $\frac{x-4}{3} = \frac{y-6}{-2} = \frac{z-2}{0}$. [5]

Given that ÷ contains the point (1, 2, 0) , show that the Cartesian equation of ÷ is $16x + 24y - 11z = 64$.

d. The z-axis meets the plane ÷ at the point P. Find the coordinates of P. [2]

e. Find the angle between the line $\frac{x-2}{3} = \frac{y+5}{4} = \frac{z}{2}$ and the plane ÷ . [5]

Markscheme

a.

in augmented matrix form $\begin{vmatrix} 1 & -3 & 1 & 3 \\ 1 & 5 & -2 & 1 \\ 0 & 16 & -6 & k \end{vmatrix}$

attempt to find a line of zeros ***(M1)***

$r_2 - r_1 \begin{vmatrix} 1 & -3 & 1 & 3 \\ 0 & 8 & -3 & -2 \\ 0 & 16 & -6 & k \end{vmatrix}$ ***(A1)***

$r_3 - 2r_2 \begin{vmatrix} 1 & -3 & 1 & 3 \\ 0 & 8 & -3 & -2 \\ 0 & 0 & 0 & k+4 \end{vmatrix}$ ***(A1)***

there is an infinite number of solutions when $k = -4$ ***R1***

there is no solution when

$k \neq -4, \ (k \in \mathbb{R})$ ***R1***

Note: Approaches other than using the augmented matrix are acceptable.

[5 marks]

b.

using $\begin{vmatrix} 1 & -3 & 1 & 3 \\ 0 & 8 & -3 & -2 \\ 0 & 0 & 0 & k+4 \end{vmatrix}$ and letting $\boldsymbol{z} = \lambda$ ***(M1)***

$8y - 3\lambda = -2$

$\Rightarrow y = \frac{3\lambda-2}{8}$ ***(A1)***

$x - 3y + z = 3$

$\Rightarrow x - \left(\frac{9\lambda-6}{8}\right) + \lambda = 3$ ***(M1)***

$\Rightarrow 8x - 9\lambda + 6 + 8\lambda = 24$

$\Rightarrow x = \frac{18+\lambda}{8}$ ***(A1)***

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{18}{8} \\ -\frac{2}{8} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{8} \\ \frac{3}{8} \\ 1 \end{pmatrix}$ ***(M1)(A1)***

$r = \begin{pmatrix} \frac{9}{4} \\ -\frac{1}{4} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix}$ ***A1***

Note: Accept equivalent answers.

[7 marks]

c.

recognition that $\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ is parallel to the plane ***(A1)***

direction normal of the plane is given by $\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 3 & 8 \\ 3 & -2 & 0 \end{vmatrix}$ ***(M1)***

$= 16\boldsymbol{i} + 24\boldsymbol{j} - 11\boldsymbol{k}$ ***A1***

Cartesian equation of the plane is given by $16x + 24y - 11z = d$ and a point which fits this equation is (1, 2, 0) ***(M1)***

$\Rightarrow 16 + 48 = d$

$d = 64$ ***A1***

hence Cartesian equation of plane is $16x + 24y - 11z = 64$ ***AG***

Note: Accept alternative methods using dot product.

[5 marks]

d. the plane crosses the z-axis when $x = y = 0$ ***(M1)***

coordinates of P are $\left(0,\ 0,\ -\frac{64}{11}\right)$ ***A1***

Note: Award ***A1*** for stating $z = -\frac{64}{11}$.

Note: Accept. $\begin{pmatrix} 0 \\ 0 \\ -\frac{64}{11} \end{pmatrix}$

[2 marks]

e. recognition that the angle between the line and the direction normal is given by:

$\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \begin{pmatrix} 16 \\ 24 \\ -11 \end{pmatrix} = \sqrt{29}\sqrt{953}\cos\theta$ where θ is the angle between the line and the normal vector ***M1A1***

$\Rightarrow 122 = \sqrt{29}\sqrt{953}\cos\theta$ ***(A1)***

$\Rightarrow \theta = 42.8^\circ$ (0.747 radians) ***(A1)***

hence the angle between the line and the plane is 90° – 42.8° = 47.2° (0.824 radians) ***A1***

[5 marks]

Note: Accept use of the formula $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$.

Examiners report

a. Many candidates were able to start this question, but only a few candidates gained full marks. Many candidates successfully used the augmented matrix in part (a) to find the correct answer. Part (b) was less successful with only a limited number of candidates using the calculator to its full effect here and with many candidates making arithmetic and algebraic errors. This was the hardest part of the question. Many candidates understood the vector techniques necessary to answer parts (c), (d) and (e) but a number made arithmetic and algebraic errors in the working.

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(a) Find the coordinates of the point A on l_1 and the point B on l_2 such that $\overrightarrow{\mathrm{AB}}$ is perpendicular to both l_1 and l_2 .

(b) Find $|\mathrm{AB}|$.

(c) Find the Cartesian equation of the plane $\prod$ which contains l_1 and does not intersect l_2 .

Markscheme

(a) on l_1 $\mathrm{A}(-3+3\lambda, -4+2\lambda, 6-2\lambda)$ ***A1***

on l_2 $l_2 : r = \begin{pmatrix} 4 \\ -7 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix}$ ***(M1)***

$\Rightarrow \mathrm{B}(4-3\mu, -7+4\mu, -3-\mu)$ ***A1***

$\overrightarrow{\mathrm{BA}} = \boldsymbol{a} - \boldsymbol{b} = \begin{pmatrix} 3\lambda+3\mu-7 \\ 2\lambda-4\mu+3 \\ -2\lambda+\mu+9 \end{pmatrix}$ ***(M1)A1***

EITHER

$\mathrm{BA} \perp l_1 \Rightarrow \mathrm{BA} \cdot \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} = 0 \Rightarrow 3\left(3\lambda+3\mu-7\right) + 2\left(2\lambda-4\mu+3\right) - 2\left(-2\lambda+\mu+9\right) = 0$ ***M1***

$\Rightarrow 17\lambda - \mu = 33$ ***A1***

$\mathrm{BA} \perp l_2 \Rightarrow \mathrm{BA} \cdot \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix} = 0 \Rightarrow -3\left(3\lambda+3\mu-7\right) + 4\left(2\lambda-4\mu+3\right) - \left(-2\lambda+\mu+9\right) = 0$ ***M1***

$\Rightarrow \lambda - 26\mu = -24$ ***A1***

solving both equations above simultaneously gives
$\lambda = 2; \mu = 1 \Rightarrow$ A(3, 0, 2), B(1, –3, –4) ***A1A1A1A1***

OR

$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 2 & -2 \\ -3 & 4 & -1 \end{vmatrix} = 6\boldsymbol{i} + 9\boldsymbol{j} + 18\boldsymbol{k}$ ***M1A1***

so $\overrightarrow{\mathrm{AB}} = p \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 3\lambda+3\mu-7 \\ 2\lambda-4\mu+3 \\ -2\lambda+\mu+9 \end{pmatrix}$ ***M1A1***

$3\lambda + 3\mu - 2p = 7$
$2\lambda - 4\mu - 3p = -3$
$-2\lambda + \mu - 6p = -9$

$\lambda = 2, \mu = 1, p = 1$ ***A1A1***

$A(-3+6, -4+4, 6-4) = (3, 0, 2)$ ***A1***

$B(4-3, -7+4, -3-1) = (1, -3, -4)$ ***A1***

[13 marks]

(b) $AB = \begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ -6 \end{pmatrix}$ ***(A1)***

$|AB| = \sqrt{(-2)^2 + (-3)^2 + (-6)^2} = \sqrt{49} = 7$ ***M1A1***

[3 marks]

(c) from (b) $2\boldsymbol{i} + 3\boldsymbol{j} + 6\boldsymbol{k}$ is normal to both lines

l_1 goes through (–3, –4, 6) $\Rightarrow \begin{pmatrix} -3 \\ -4 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} = 18$ ***M1A1***

hence, the Cartesian equation of the plane through l_1 , but not l_2 , is $2x + 3y + 6z = 18$ ***A1***

[3 marks]

Total [19 marks]

Examiners report

There were a lot of arithmetic errors in the treatment of this question, even though it was apparent that many students did understand the methods involved. In (a) many students failed to realise that $\overrightarrow{AB}$ should be a multiple of the cross product of the two direction vectors, rather than the cross product itself, and many students failed to give the final answer as coordinates.

The angle between the vector $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ and the vector $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j} + m\boldsymbol{k}$ is 30° .

Find the values of *m*.

Markscheme

$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$ ***(M1)***

$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ m \end{pmatrix} = 7 + 3m$ ***A1***

$|\boldsymbol{a}| = \sqrt{14}$ $\quad |\boldsymbol{b}| = \sqrt{13 + m^2}$ ***A1***

$|\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta = \sqrt{14}\sqrt{13 + m^2} \cos 30^\circ$

$7 + 3m = \sqrt{14}\sqrt{13 + m^2} \cos 30^\circ$ ***A1***

$m = 2.27$, $m = 25.7$ ***A1A1***

[6 marks]

Examiners report

Many candidates gained the first 4 marks by obtaining the equation, in unsimplified form, satisfied by *m* but then made mistakes in simplifying and solving this equation.

Find the angle between the lines $\frac{x-1}{2} = 1 - y = 2z$ and $x = y = 3z$.

Markscheme

consider a vector parallel to each line,

e.g. $\boldsymbol{u} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$ ***A1A1***

let θ be the angle between the lines

$\cos\theta = \frac{|\boldsymbol{u}\times\boldsymbol{v}|}{|\boldsymbol{u}||\boldsymbol{v}|} = \frac{|12-6+1|}{\sqrt{21}\sqrt{19}}$ ***M1A1***

$= \frac{7}{\sqrt{21}\sqrt{19}} = 0.350...$ ***(A1)***

so $\theta = 69.5 \left(\text{or } 1.21 \text{ rad or } \arccos\left(\frac{7}{\sqrt{21}\sqrt{19}}\right)\right)$ ***A1 N4***

Note: Allow FT from incorrect reasonable vectors.

[6 marks]

Examiners report

Most students knew how to find the angle between two vectors, although many could not find the correct two direction vectors.

A curve is defined $x^2 - 5xy + y^2 = 7$.

a. Show that $\frac{dy}{dx} = \frac{5y-2x}{2y-5x}$. [3]

b. Find the equation of the normal to the curve at the point $(6,\ 1)$. [4]

c. Find the distance between the two points on the curve where each tangent is parallel to the line $y = x$. [8]

Markscheme

a. attempt at implicit differentiation ***M1***

$$2x - 5x\frac{dy}{dx} - 5y + 2y\frac{dy}{dx} = 0$$ ***A1A1***

Note: ***A1*** for differentiation of $x^2 - 5xy$, ***A1*** for differentiation of y^2 and 7.

$2x - 5y + \frac{dy}{dx}(2y - 5x) = 0$

$\frac{dy}{dx} = \frac{5y-2x}{2y-5x}$ ***AG***

[3 marks]

b. $\frac{dy}{dx} = \frac{5\times1-2\times6}{2\times1-5\times6} = \frac{1}{4}$ ***A1***

gradient of normal $= -4$ ***A1***

equation of normal $y = -4x + c$ ***M1***

substitution of $(6,\ 1)$

$y = -4x + 25$ ***A1***

Note: Accept $y - 1 = -4(x - 6)$

[4 marks]

c. setting $\frac{5y-2x}{2y-5x} = 1$ ***M1***

$y = -x$ ***A1***

substituting into original equation ***M1***

$x^2 + 5x^2 + x^2 = 7$ ***(A1)***

$7x^2 = 7$

$x = \pm1$ ***A1***

points $(1,\ -1)$ and $(-1,\ 1)$ ***(A1)***

distance $= \sqrt{8}$ $\left(= 2\sqrt{2}\right)$ ***(M1)A1***

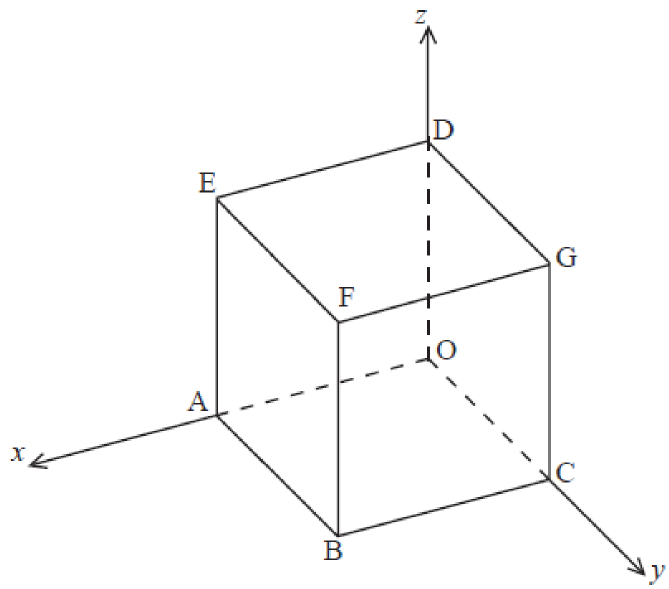
[8 marks]

Total [15 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

The diagram shows a cube OABCDEFG.



Let O be the origin, (OA) the x-axis, (OC) the y-axis and (OD) the z-axis.

Let M, N and P be the midpoints of [FG], [DG] and [CG], respectively.

The coordinates of F are (2, 2, 2).

(a) Find the position vectors $\overrightarrow{\text{OM}}$, $\overrightarrow{\text{ON}}$ and $\overrightarrow{\text{OP}}$ in component form.

(b) Find $\overrightarrow{\text{MP}} \times \overrightarrow{\text{MN}}$.

(c) **Hence**,

(i) calculate the area of the triangle MNP;

(ii) show that the line (AG) is perpendicular to the plane MNP;

(iii) find the equation of the plane MNP.

(d) Determine the coordinates of the point where the line (AG) meets the plane MNP.

Markscheme

(a) $\overrightarrow{\text{OM}} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \overrightarrow{\text{ON}} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ and $\overrightarrow{\text{OP}} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ ***A1A1A1***

[3 marks]

(b) $\overrightarrow{\text{MP}} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$ and $\overrightarrow{\text{MN}} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ ***A1A1***

$\overrightarrow{\text{MP}} \times \overrightarrow{\text{MN}} = \begin{pmatrix} i & j & k \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ ***(M1)A1***

[4 marks]

(c) (i) area of MNP $= \frac{1}{2}\left|\overrightarrow{\text{MP}} \times \overrightarrow{\text{MN}}\right|$ ***M1***

$= \frac{1}{2}\left|\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right|$

$= \frac{\sqrt{3}}{2}$ ***A1***

(ii) $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \overrightarrow{OG} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$

$\overrightarrow{AG} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$ ***A1***

since $\overrightarrow{AG} = 2(\overrightarrow{MP} \times \overrightarrow{MN})$ AG is perpendicular to MNP ***R1***

(iii) $r \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ ***M1A1***

$r \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 3$ (accept $-x + y + z = 3$) ***A1***

[7 marks]

(d) $r = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$ ***A1***

$\begin{pmatrix} 2-2\lambda \\ 2\lambda \\ 2\lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 3$ ***M1A1***

$-2 + 2\lambda + 2\lambda + 2\lambda = 3$

$\lambda = \frac{5}{6}$ ***A1***

$r = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \frac{5}{6}\begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$ ***M1***

coordinates of point $\left(\frac{1}{3}, \frac{5}{3}, \frac{5}{3}\right)$ ***A1***

[6 marks]

Total [20 marks]

Examiners report

This was the most successfully answered question in part B, with many candidates achieving full marks. There were a few candidates who misread the question and treated the cube as a unit cube. The most common errors were either algebraic or arithmetic mistakes. A variety of notation forms were seen but in general were used consistently. In a few cases, candidates failed to show all the work or set it properly.

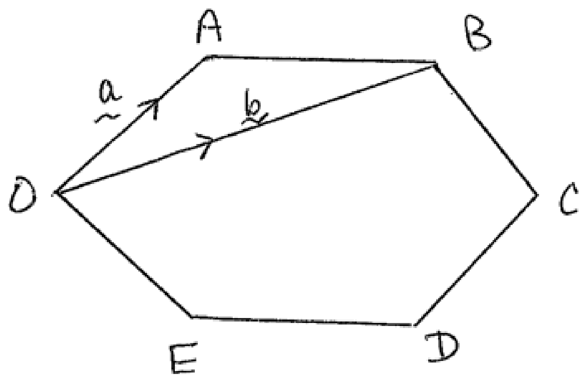
OABCDE is a regular hexagon and ***a*** , ***b*** denote respectively the position vectors of A, B with respect to O.

a. Show that OC = 2AB . [2]

b. Find the position vectors of C, D and E in terms of ***a*** and ***b*** . [7]

Markscheme

a.



$\mathrm{OC} = \mathrm{AB} + \mathrm{OA}\cos 60 + \mathrm{BC}\cos 60$ ***M1***

$= \mathrm{AB} + \mathrm{AB} \times \frac{1}{2} + \mathrm{AB} \times \frac{1}{2}$ ***A1***

$= 2\mathrm{AB}$ ***AG***

[2 marks]

b. $\overrightarrow{\mathrm{OC}} = 2\overrightarrow{\mathrm{AB}} = 2(\boldsymbol{b} - \boldsymbol{a})$ ***M1A1***

$\overrightarrow{\mathrm{OD}} = \overrightarrow{\mathrm{OC}} + \overrightarrow{\mathrm{CD}}$ ***M1***

$= \overrightarrow{\mathrm{OC}} + \overrightarrow{\mathrm{AO}}$ ***A1***

$= 2\boldsymbol{b} - 2\boldsymbol{a} - \boldsymbol{a} = 2\boldsymbol{b} - 3\boldsymbol{a}$ ***A1***

$\overrightarrow{\mathrm{OE}} = \overrightarrow{\mathrm{BC}}$ ***M1***

$= 2\boldsymbol{b} - 2\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{b} - 2\boldsymbol{a}$ ***A1***

[7 marks]

Examiners report

a. [N/A]

b. [N/A]

Two planes Π_1 and Π_2 have equations $2x + y + z = 1$ and $3x + y - z = 2$ respectively.

a. Find the vector equation of L, the line of intersection of Π_1 and Π_2. [6]

b. Show that the plane Π_3 which is perpendicular to Π_1 and contains L, has equation $x - 2z = 1$. [4]

c. The point P has coordinates $(-2, 4, 1)$, the point Q lies on Π_3 and PQ is perpendicular to Π_2. Find the coordinates of Q. [6]

Markscheme

a. (a) **METHOD 1**

solving simultaneously (gdc) ***(M1)***

$x = 1 + 2z;\ y = -1 - 5z$ ***A1A1***

$L: \boldsymbol{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$ ***A1A1A1***

Note: 1st ***A1*** is for ***r*** =.

[6 marks]

METHOD 2

direction of line = $\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix}$ (last two rows swapped) ***M1***

= 2***i*** − 5***j*** + ***k*** ***A1***

putting *z* = 0, a point on the line satisfies $2x + y = 1,\ 3x + y = 2$ ***M1***

i.e. (1, −1, 0) ***A1***

the equation of the line is

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$ ***A1A1***

Note: Award ***A0A1*** if $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is missing.

[6 marks]

b. $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$ ***M1***

= 6***i*** − 12***k*** ***A1***

hence, ***n*** = ***i*** − 2***k***

$\boldsymbol{n} \cdot \boldsymbol{a} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1$ ***M1A1***

therefore ***r*** · ***n*** = ***a*** · ***n*** ⇒ $x - 2z = 1$ ***AG***

[4 marks]

c. **METHOD 1**

P = (−2, 4, 1), Q = $(x,\ y,\ z)$

$\overrightarrow{\mathrm{PQ}} = \begin{pmatrix} x+2 \\ y-4 \\ z-1 \end{pmatrix}$ ***A1***

$\overrightarrow{\mathrm{PQ}}$ is perpendicular to $3x + y - z = 2$

⇒ $\overrightarrow{\mathrm{PQ}}$ is parallel to 3***i*** + ***j*** − ***k*** ***R1***

$\Rightarrow x + 2 = 3t;\ y - 4 = t;\ z - 1 = -t$ ***A1***

$1 - z = t \Rightarrow x + 2 = 3 - 3z \Rightarrow x + 3z = 1$ ***A1***

solving simultaneously $x + 3z = 1;\ x - 2z = 1$ ***M1***

$5z = 0 \Rightarrow z = 0;\ x = 1,\ y = 5$ ***A1***

hence, Q = (1, 5, 0)

[6 marks]

METHOD 2

Line passing through PQ has equation

$\mathbf{r} = \begin{matrix} -2 \\ 4 \\ 1 \end{matrix} + t \begin{matrix} 3 \\ 1 \\ -1 \end{matrix}$ ***M1A1***

Meets π_3 when:

$-2 + 3t - 2(1 - t) = 1$ ***M1A1***

$t = 1$ ***A1***

Q has coordinates (1, 5, 0) ***A1***

[6 marks]

Examiners report

a. Candidates generally attempted this question but with varying degrees of success. Although (a) was answered best of all the parts, quite a few did not use correct notation to designate the vector equation of a line, i.e., ***r*** =, or its equivalent. In (b) some candidates incorrectly assumed the result and worked the question from there. In (c) some candidates did not understand the necessary relationships to make a meaningful attempt.

b. Candidates generally attempted this question but with varying degrees of success. Although (a) was answered best of all the parts, quite a few did not use correct notation to designate the vector equation of a line, i.e., ***r*** =, or its equivalent. In (b) some candidates incorrectly assumed the result and worked the question from there. In (c) some candidates did not understand the necessary relationships to make a meaningful attempt.

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Consider the two planes

$\pi_1 : 4x + 2y - z = 8$

$\pi_2 : x + 3y + 3z = 3.$

Find the angle between π_1 and π_2, giving your answer correct to the nearest degree.

Markscheme

$n_1 = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$ and $n_2 = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$ ***(A1)(A1)***

use of $\cos\theta = \frac{n_1 \bullet n_2}{|n_1||n_2|}$ ***(M1)***

$\cos\theta = \frac{7}{\sqrt{21}\sqrt{19}} \left(= \frac{7}{\sqrt{399}}\right)$ ***(A1)(A1)***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

$\theta = 69^\circ$ ***A1***

Note: Award ***A1*** for 111°.

[6 marks]

Examiners report

Reasonably well answered. A large number of candidates did not express their final answer correct to the nearest degree.

A ray of light coming from the point (−1, 3, 2) is travelling in the direction of vector $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ and meets the plane $\pi : x + 3y + 2z - 24 = 0$.

Find the angle that the ray of light makes with the plane.

Markscheme

The normal vector to the plane is $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$. ***(A1)***

EITHER

θ is the angle between the line and the normal to the plane.

$\cos\theta = \frac{\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}}{\sqrt{14}\sqrt{21}} = \frac{3}{\sqrt{14}\sqrt{21}} = \left(\frac{3}{7\sqrt{6}}\right)$ ***(M1)A1A1***

$\Rightarrow \theta = 79.9^\circ \ (= 1.394\ldots)$ ***A1***

The required angle is 10.1° (= 0.176) ***A1***

OR

ϕ is the angle between the line and the plane.

$\sin\phi = \frac{\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}}{\sqrt{14}\sqrt{21}} = \frac{3}{\sqrt{14}\sqrt{21}}$ ***(M1)A1A1***

ϕ = 10.1° (= 0.176) ***A2***

[6 marks]

Examiners report

On the whole this question was well answered. Some candidates failed to find the complementary angle when using the formula with cosine.

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are such that $\boldsymbol{a} = (3\cos\theta + 6)\boldsymbol{i} + 7\boldsymbol{j}$ and $\boldsymbol{b} = (\cos\theta - 2)\boldsymbol{i} + (1 + \sin\theta)\boldsymbol{j}$.

Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular,

a. show that $3\sin^2\theta - 7\sin\theta + 2 = 0$; [3]

b. find the smallest possible positive value of θ. [3]

Markscheme

a. attempting to form $(3\cos\theta + 6)(\cos\theta - 2) + 7(1 + \sin\theta) = 0$ ***M1***

$3\cos^2\theta - 12 + 7\sin\theta + 7 = 0$ ***A1***

$3\left(1 - \sin^2\theta\right) + 7\sin\theta - 5 = 0$ ***M1***

$3\sin^2\theta - 7\sin\theta + 2 = 0$ ***AG***

[3 marks]

b. attempting to solve algebraically (including substitution) or graphically for $\sin\theta$ ***(M1)***

$\sin\theta = \frac{1}{3}$ ***(A1)***

$\theta = 0.340\ (= 19.5^\circ)$ ***A1***

[3 marks]

Examiners report

a. Part (a) was very well done. Most candidates were able to use the scalar product and $\cos^2\theta = 1 - \sin^2\theta$ to show the required result.

b. Part (b) was reasonably well done. A few candidates confused ‘smallest possible positive value’ with a minimum function value. Some candidates gave $\theta = 0.34$ as their final answer.

Given that $\boldsymbol{a} = 2\sin\theta\boldsymbol{i} + (1 - \sin\theta)\,\boldsymbol{j}$, find the value of the acute angle θ, so that $\boldsymbol{a}$ is perpendicular to the line $x + y = 1$.

Markscheme

direction vector for line $= \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or any multiple ***A1***

$\begin{pmatrix} 2\sin\theta \\ 1 - \sin\theta \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$ ***M1***

$2\sin\theta - 1 + \sin\theta = 0$ ***A1***

Note: Allow ***FT*** on candidate’s direction vector just for line above only.

$3\sin\theta = 1$

$\sin\theta = \frac{1}{3}$ ***A1***

$\theta = 0.340$ or 19.5 ***A1***

Note: A coordinate geometry method using perpendicular gradients is acceptable.

[5 marks]

Examiners report

A variety of approaches were seen, either using a scalar product of vectors, or based on the rule for perpendicular gradients of lines. The main problem encountered in the first approach was in the choice of the correct vector direction for the line.

A line L_1 has equation $\boldsymbol{r} = \begin{pmatrix} -5 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$.

A line L_2 passing through the origin intersects L_1 and is perpendicular to L_1.

(a) Find a vector equation of L_2.

(b) Determine the shortest distance from the origin to L_1.

Markscheme

(a) **METHOD 1**

for P on L_1, $\overrightarrow{\text{OP}} = \begin{pmatrix} -5-\lambda \\ -3+2\lambda \\ 2+2\lambda \end{pmatrix}$

require $\begin{pmatrix} -5-\lambda \\ -3+2\lambda \\ 2+2\lambda \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 0$ ***M1***

$5+\lambda-6+4\lambda+4+4\lambda = 0$ (or equivalent) ***A1***

$\lambda = -\frac{1}{3}$ ***A1***

$\therefore \overrightarrow{\text{OP}} = \begin{pmatrix} -\frac{14}{3} \\ -\frac{11}{3} \\ \frac{4}{3} \end{pmatrix}$ ***A1***

$L_2 : \boldsymbol{r} = \mu \begin{pmatrix} -14 \\ -11 \\ 4 \end{pmatrix}$ ***A1***

Note: Do not award the final ***A1*** if $\boldsymbol{r} =$ is not seen.

[5 marks]

METHOD 2

Calculating either $\left|\overrightarrow{\text{OP}}\right|$ or $\left|\overrightarrow{\text{OP}}\right|^2$ *eg*

$\left|\overrightarrow{\text{OP}}\right| = \sqrt{(-5-\lambda)^2 + (-3+2\lambda)^2 + (2+2\lambda)^2}$ ***A1***

$= \sqrt{9\lambda^2 + 6\lambda + 38}$

Solving either $\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\left|\overrightarrow{\text{OP}}\right|\right) = 0$ or $\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\left|\overrightarrow{\text{OP}}\right|^2\right) = 0$ for λ. ***M1***

$\lambda = -\frac{1}{3}$ ***A1***

$\overrightarrow{\text{OP}} = \begin{pmatrix} -\frac{14}{3} \\ -\frac{11}{3} \\ \frac{4}{3} \end{pmatrix}$ ***A1***

$L_2 : \boldsymbol{r} = \mu \begin{pmatrix} -14 \\ -11 \\ 4 \end{pmatrix}$ ***A1***

Note: Do not award the final ***A1*** if $\boldsymbol{r} =$ is not seen.

[5 marks]

(b) **METHOD 1**

$\left|\overrightarrow{\mathrm{OP}}\right| = \sqrt{\left(-\frac{14}{3}\right)^2 + \left(-\frac{11}{3}\right)^2 + \left(\frac{4}{3}\right)^2}$ ***(M1)***

$= 6.08 \left(= \sqrt{37}\right)$ ***A1***

METHOD 2

$$\text{shortest distance} = \frac{\left|\begin{pmatrix}-1\\2\\2\end{pmatrix} \times \begin{pmatrix}-5\\-3\\2\end{pmatrix}\right|}{\left|\begin{pmatrix}-1\\2\\2\end{pmatrix}\right|}$$ ***(M1)***

$= \frac{|10i+8j+13k|}{\sqrt{1+4+4}}$

$= 6.08 \left(= \sqrt{37}\right)$ ***A1***

[2 marks]

Total [7 marks]

Examiners report

Part (a) was not well done. Most candidates recognised the need to calculate a scalar product. Some candidates made careless sign or arithmetic errors when solving for λ. A few candidates neglected to express their final answer in the form '$\boldsymbol{r}$ ='.

Candidates who were successful in answering part (a) generally answered part (b) correctly. The large majority of successful candidates calculated $\left|\overrightarrow{\mathrm{OP}}\right|$.

Port A is defined to be the origin of a set of coordinate axes and port B is located at the point (70, 30), where distances are measured in kilometres.

A ship S_1 sails from port A at 10:00 in a straight line such that its position t hours after 10:00 is given by $r = t\begin{pmatrix}10\\20\end{pmatrix}$.

A speedboat S_2 is capable of three times the speed of S_1 and is to meet S_1 by travelling the shortest possible distance. What is the latest time that S_2 can leave port B?

Markscheme

METHOD 1

equation of journey of ship S_1

$r_1 = t\begin{pmatrix}10\\20\end{pmatrix}$

equation of journey of speedboat S_2 ,setting off k minutes later

$r_2 = \begin{pmatrix}70\\30\end{pmatrix} + (t-k)\begin{pmatrix}-60\\30\end{pmatrix}$ ***M1A1A1***

Note: Award ***M1*** for perpendicular direction, ***A1*** for speed, ***A1*** for change in parameter (*e.g.* by using $t-k$ or T, k being the time difference between the departure of the ships).

solve $t\begin{pmatrix}10\\20\end{pmatrix}=\begin{pmatrix}70\\30\end{pmatrix}+(t-k)\begin{pmatrix}-60\\30\end{pmatrix}$ ***(M1)***

Note: ***M*** mark is for equating their two expressions.

$10t=70-60t+60k$

$20t=30+30t-30k$ ***M1***

Note: ***M*** mark is for obtaining two equations involving two different parameters.

$7t-6k=7$

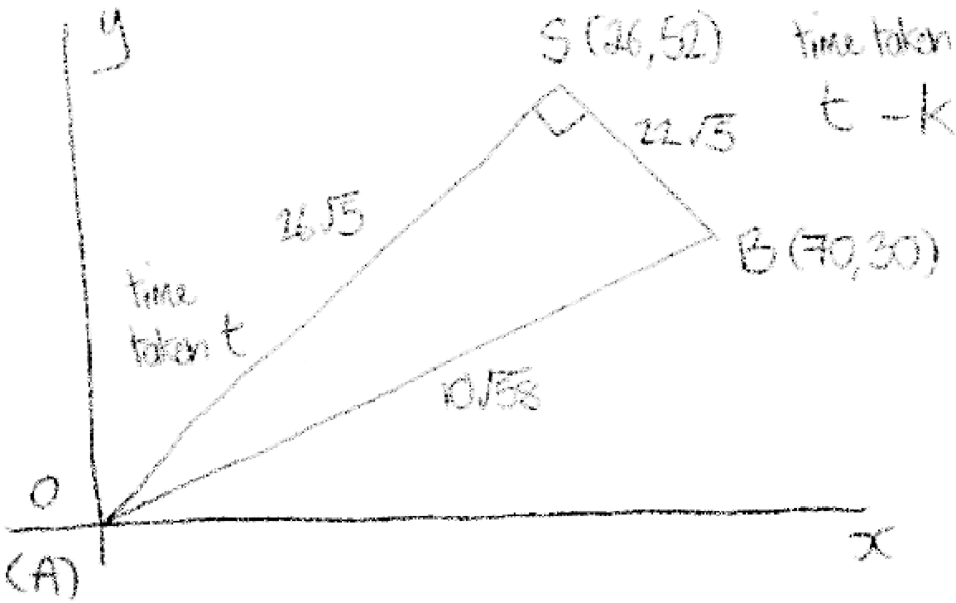
$-t+3k=3$

$k=\frac{28}{15}$ ***A1***

latest time is 11:52 ***A1***

[7 marks]

METHOD 2



$\mathrm{SB}=22\sqrt{5}$ ***M1A1***
(by perpendicular distance)

$\mathrm{SA}=26\sqrt{5}$ ***M1A1***
(by Pythagoras or coordinates)

$t=\frac{26\sqrt{5}}{10\sqrt{5}}$ **A1**

$t-k=\frac{22\sqrt{5}}{30\sqrt{5}}$ ***A1***

$k=\frac{28}{15}$ leading to latest time 11:52 ***A1***

[7 marks]

Examiners report

Few candidates managed to make progress on this question. Many candidates did not attempt the problem and many that did make an attempt failed to draw a diagram that would have allowed them to make further progress. There were a variety of possible solution techniques but candidates seemed unable to interpret the equation of a straight line written in vector form or find a perpendicular direction. This meant that it was very difficult for meaningful progress to be made towards a solution.

The coordinates of points A, B and C are given as $(5,\ -2,\ 5)$, $(5,\ 4,\ -1)$ and $(-1,\ -2,\ -1)$ respectively.

a. Show that AB = AC and that $\mathrm{B\hat{A}C} = 60^{\circ}$. [4]

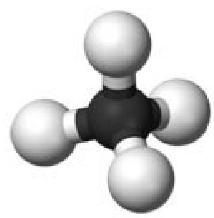
b. Find the Cartesian equation of Π, the plane passing through A, B, and C. [4]

c(i)(ii). (i) Find the Cartesian equation of Π_1 , the plane perpendicular to (AB) passing through the midpoint of [AB] . [4]

(ii) Find the Cartesian equation of Π_2 , the plane perpendicular to (AC) passing through the midpoint of [AC].

d. Find the vector equation of L , the line of intersection of Π_1 and Π_2 , and show that it is perpendicular to Π . [3]

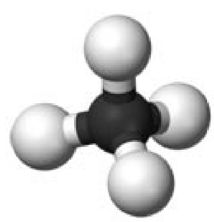
e. A methane molecule consists of a carbon atom with four hydrogen atoms symmetrically placed around it in three dimensions. [3]



The positions of the centres of three of the hydrogen atoms are A, B and C as given. The position of the centre of the fourth hydrogen atom is D.

Using the fact that $\mathrm{AB} = \mathrm{AD}$, show that the coordinates of one of the possible positions of the fourth hydrogen atom is $(-1,\ 4,\ 5)$.

f. A methane molecule consists of a carbon atom with four hydrogen atoms symmetrically placed around it in three dimensions. [6]



The positions of the centres of three of the hydrogen atoms are A, B and C as given. The position of the centre of the fourth hydrogen atom is D.

Letting D be $(-1,\ 4,\ 5)$, show that the coordinates of G, the position of the centre of the carbon atom, are $(2,\ 1,\ 2)$. Hence calculate $\mathrm{D\hat{G}A}$, the bonding angle of carbon.

Markscheme

a. $\overrightarrow{\mathrm{AB}} = \begin{pmatrix} 0 \\ 6 \\ -6 \end{pmatrix} \Rightarrow \mathrm{AB} = \sqrt{72}$ ***A1***

$\overrightarrow{\mathrm{AC}} = \begin{pmatrix} -6 \\ 0 \\ -6 \end{pmatrix} \Rightarrow \mathrm{AC} = \sqrt{72}$ ***A1***

so they are the same ***AG***

$\overrightarrow{\mathrm{AB}} \cdot \overrightarrow{\mathrm{AC}} = 36 = \left(\sqrt{72}\right)\left(\sqrt{72}\right)\cos\theta$ ***(M1)***

$\cos\theta = \frac{36}{\left(\sqrt{72}\right)\left(\sqrt{72}\right)} = \frac{1}{2} \Rightarrow \theta = 60^\circ$ ***A1AG***

Note: Award ***M1A1*** if candidates find BC and claim that triangle ABC is equilateral.

[4 marks]

b. **METHOD 1**

$\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 6 & -6 \\ -6 & 0 & -6 \end{vmatrix} = -36\boldsymbol{i} + 36\boldsymbol{j} + 36\boldsymbol{k}$ ***(M1)A1***

equation of plane is $x - y - z = k$ ***(M1)***

goes through A, B or C $\Rightarrow x - y - z = 2$ ***A1***

[4 marks]

METHOD 2

$x + by + cz = d$ (or similar) ***M1***

$5 - 2b + 5c = d$

$5 + 4b - c = d$ ***A1***

$-1 - 2b - c = d$

solving simultaneously ***M1***

$b = -1,\ c = -1,\ d = 2$

so $x - y - z = 2$ ***A1***

[4 marks]

c(i)(ii) midpoint is $(5,\ 1,\ 2)$, so equation of Π_1 is $y - z = -1$ ***A1A1***

(ii) midpoint is $(2,\ -2,\ 2)$, so equation of Π_2 is $x + z = 4$ ***A1A1***

Note: In each part, award ***A1*** for midpoint and ***A1*** for the equation of the plane.

[4 marks]

d. **EITHER**

solving the two equations above ***M1***

$L: r = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ ***A1***

OR

L has the direction of the vector product of the normal vectors to the planes Π_1 and Π_2 ***(M1)***

$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$

(or its opposite) ***A1***

THEN

direction is $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ as required ***R1***

[3 marks]

e. D is of the form $(4-\lambda,\ -1+\lambda,\ \lambda)$ ***M1***

$(1+\lambda)^2+(-1-\lambda)^2+(5-\lambda)^2=72$ ***M1***
$3\lambda^2-6\lambda-45=0$
$\lambda=5$ or $\lambda=-3$ ***A1***
$\mathrm{D}(-1,\ 4,\ 5)$ ***AG***
Note: Award ***M0M0A0*** if candidates just show that $\mathrm{D}(-1,\ 4,\ 5)$ satisfies $\mathrm{AB}=\mathrm{AD}$;
Award ***M1M1A0*** if candidates also show that D is of the form $(4-\lambda,\ -1+\lambda,\ \lambda)$

[3 marks]

f. **EITHER**

G is of the form $(4-\lambda,\ -1+\lambda,\ \lambda)$ and $\mathrm{DG}=\mathrm{AG}$, BG or CG ***M1***
e.g. $(1+\lambda)^2+(-1-\lambda)^2+(5-\lambda)^2=(5-\lambda)^2+(5-\lambda)^2+(5-\lambda)^2$ ***M1***
$(1+\lambda)^2=(5-\lambda)^2$
$\lambda=2$ ***A1***
$\mathrm{G}(2,\ 1,\ 2)$ ***AG***
OR
G is the centre of mass (barycentre) of the regular tetrahedron ABCD ***(M1)***
$\mathrm{G}\left(\frac{5+5+(-1)+(-1)}{4},\ \frac{-2+4+(-2)+4}{4},\ \frac{5+(-1)+(-1)+5}{4}\right)$ ***M1A1***
THEN
Note: the following part is independent of previous work and candidates may use ***AG*** to answer it (here it is possible to award ***M0M0A0A1M1A1)***

$\overrightarrow{GD}=\begin{pmatrix}-3\\3\\3\end{pmatrix}$ and $\overrightarrow{GA}=\begin{pmatrix}3\\-3\\3\end{pmatrix}$ ***A1***

$\cos\theta=\frac{-9}{\left(3\sqrt{3}\right)\left(3\sqrt{3}\right)}=-\frac{1}{3}\Rightarrow\theta=109^\circ$ (or 1.91 radians) ***M1A1***
[6 marks]

Examiners report

a. Parts (a) and (b) were well attempted with many candidates achieving full or nearly full marks in these parts. Surprisingly, many candidates were not able to find the coordinates of the midpoints in part (c) leading to incorrect equations of the planes and, in many cases this affected the performance in part (d). Very few answered part (d) correctly. In parts (e) and (f) very few candidates made good attempts of using the ‘show that’ procedure and used a parametric approach. In most attempts, candidates simply verified the condition using the answer given. In a few cases, candidates noticed that G is the barycentre of [ABCD] and found its coordinates successfully. Some candidates were also successful in determining the bounding angle using the information given. Unfortunately, many candidates instead of finding the angle simply quoted the result.

b. Parts (a) and (b) were well attempted with many candidates achieving full or nearly full marks in these parts. Surprisingly, many candidates were not able to find the coordinates of the midpoints in part (c) leading to incorrect equations of the planes and, in many cases this affected the performance in part (d). Very few answered part (d) correctly. In parts (e) and (f) very few candidates made good attempts of using the ‘show that’ procedure and used a parametric approach. In most attempts, candidates simply verified the condition using the answer given. In a few cases, candidates noticed that G is the barycentre of [ABCD] and found its coordinates successfully. Some candidates were also successful in determining the bounding angle using the information given. Unfortunately, many candidates instead of finding the angle simply quoted the result.

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The equations of the lines L_1 and L_2 are

$$L_1 : r_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$L_2 : r_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}.$

a. Show that the lines L_1 and L_2 are skew. [4]

b. Find the acute angle between the lines L_1 and L_2. [4]

c. (i) Find a vector perpendicular to both lines. [10]

(ii) Hence determine an equation of the line L_3 that is perpendicular to both L_1 and L_2 and intersects both lines.

Markscheme

a.

L_1 and L_2 are not parallel, since $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \neq k \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}$ ***R1***

if they meet, then $1 - \lambda = 1 + 2\mu$ and $2 + \lambda = 2 + \mu$ ***M1***

solving simultaneously $\Rightarrow \lambda = \mu = 0$ ***A1***

$2 + 2\lambda = 4 + 6\mu \Rightarrow 2 \neq 4$ contradiction, ***R1***

so lines are skew ***AG***

Note: Do not award the second ***R1*** if their values of parameters are incorrect.

[4 marks]

b.

$\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} (= 11) = \sqrt{6}\sqrt{41}\cos\theta$ ***M1A1***

$\cos\theta = \frac{11}{\sqrt{246}}$ ***(A1)***

$\theta = 45.5^\circ \quad (0.794 \text{ radians})$ ***A1***

[4 marks]

c.

(i) $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 6-2 \\ 4+6 \\ -1-2 \end{pmatrix}$ ***(M1)***

$= \begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix} = 4\mathbf{i} + 10\mathbf{j} - 3\mathbf{k}$ ***A1***

(ii) **METHOD 1**

let P be the intersection of L_1 and L_3

let Q be the intersection of L_2 and L_3

$\overrightarrow{\mathrm{OP}} = \begin{pmatrix} 1-\lambda \\ 2+\lambda \\ 2+2\lambda \end{pmatrix} \overrightarrow{\mathrm{OQ}} = \begin{pmatrix} 1+2\mu \\ 2+\mu \\ 4+6\mu \end{pmatrix}$ ***M1***

therefore $\overrightarrow{\mathrm{PQ}} = \overrightarrow{\mathrm{OQ}} - \overrightarrow{\mathrm{OP}} = \begin{pmatrix} 2\mu+\lambda \\ \mu-\lambda \\ 2+6\mu-2\lambda \end{pmatrix}$ ***M1A1***

$$\begin{pmatrix} 2\mu+\lambda \\ \mu-\lambda \\ 2+6\mu-2\lambda \end{pmatrix} = t\begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$$ ***M1***

$2\mu+\lambda-4t=0$
$\mu-\lambda-10t=0$
$6\mu-2\lambda+3t=-2$

solving simultaneously ***(M1)***

$\lambda=\frac{32}{125}\quad(0.256),\ \mu=-\frac{28}{125}\quad(-0.224)$ ***A1***

Note: Award ***A1*** for either correct λ or μ.

EITHER

therefore $\overrightarrow{\mathrm{OP}}=\begin{pmatrix} 1-\lambda \\ 2+\lambda \\ 2+2\lambda \end{pmatrix}=\begin{pmatrix} \frac{93}{125} \\ \frac{282}{125} \\ \frac{314}{125} \end{pmatrix}=\begin{pmatrix} 0.744 \\ 2.256 \\ 2.512 \end{pmatrix}$ ***A1***

therefore $L_3: r_3=\begin{pmatrix} 0.744 \\ 2.256 \\ 2.512 \end{pmatrix}+\alpha\begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$ ***A1***

OR

therefore $\overrightarrow{\mathrm{OQ}}=\begin{pmatrix} 1+2\mu \\ 2+\mu \\ 4+6\mu \end{pmatrix}=\begin{pmatrix} \frac{69}{125} \\ \frac{222}{125} \\ \frac{332}{125} \end{pmatrix}=\begin{pmatrix} 0.552 \\ 1.776 \\ 2.656 \end{pmatrix}$ ***A1***

therefore $L_3: r_3=\begin{pmatrix} 0.552 \\ 1.776 \\ 2.656 \end{pmatrix}+\alpha\begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$ ***A1***

Note: Allow position vector(s) to be expressed in decimal or fractional form.

METHOD 2

$$L_3: r_3=\begin{pmatrix} a \\ b \\ c \end{pmatrix}+t\begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$$

forming two equations as intersections with L_1 and L_2

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}+t_1\begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}+\lambda\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}+t_2\begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}+\mu\begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}$$ ***M1A1A1***

Note: Only award ***M1A1A1*** if two different parameters $t_1,\ t_2$ used.

attempting to solve simultaneously ***M1***

$\lambda=\frac{32}{125}\quad(0.256),\ \mu=-\frac{28}{125}\quad(-0.224)$ ***A1***

Note: Award ***A1*** for either correct λ or μ.

EITHER

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0.552 \\ 1.776 \\ 2.656 \end{pmatrix}$$ ***A1***

therefore $L_3 : r_3 = \begin{pmatrix} 0.552 \\ 1.776 \\ 2.656 \end{pmatrix} + t\begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$ ***A1A1***

OR

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0.744 \\ 2.256 \\ 2.512 \end{pmatrix}$$ ***A1***

therefore $L_3 : r_3 = \begin{pmatrix} 0.744 \\ 2.256 \\ 2.512 \end{pmatrix} + t\begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix}$ ***A1A1***

Note: Allow position vector(s) to be expressed in decimal or fractional form.

10 marks

Total [18 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

Find the acute angle between the planes with equations $x + y + z = 3$ and $2x - z = 2$.

Markscheme

$\boldsymbol{n}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{n}_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ ***(A1)(A1)***

EITHER

$\theta = \arccos\left(\frac{n_1 \bullet n_2}{|n_1||n_2|}\right) \left(\cos\theta = \frac{n_1 \bullet n_2}{|n_1||n_2|}\right)$ ***(M1)***

$= \arccos\left(\frac{2+0-1}{\sqrt{3}\sqrt{5}}\right) \left(\cos\theta = \frac{2+0-1}{\sqrt{3}\sqrt{5}}\right)$ ***(A1)***

$= \arccos\left(\frac{1}{\sqrt{15}}\right) \left(\cos\theta = \frac{1}{\sqrt{15}}\right)$

OR

$\theta = \arcsin\left(\frac{|n_1 \times n_2|}{|n_1||n_2|}\right) \left(\sin\theta = \frac{|n_1 \times n_2|}{|n_1||n_2|}\right)$ ***(M1)***

$= \arcsin\left(\frac{\sqrt{14}}{\sqrt{3}\sqrt{5}}\right) \left(\sin\theta = \frac{\sqrt{14}}{\sqrt{3}\sqrt{5}}\right)$ ***(A1)***

$= \arcsin\left(\frac{\sqrt{14}}{\sqrt{15}}\right) \left(\sin\theta = \frac{\sqrt{14}}{\sqrt{15}}\right)$

THEN

$= 75.0^{\circ}$ (or 1.31) ***A1***

[5 marks]

Examiners report

[N/A]

The planes $2x + 3y - z = 5$ and $x - y + 2z = k$ intersect in the line $5x + 1 = 9 - 5y = -5z$.

Find the value of k .

Markscheme

point on line is $x = \frac{-1-5\lambda}{5}$, $y = \frac{9+5\lambda}{5}$, $z = \lambda$ or similar ***M1A1***

Note: Accept use of point on the line or elimination of one of the variables using the equations of the planes

$\frac{-1-5\lambda}{5} - \frac{9+5\lambda}{5} + 2\lambda = k$ ***M1A1***

Note: Award ***M1A1*** if coordinates of point and equation of a plane is used to obtain linear equation in *k* or equations of the line are used in combination with equation obtained by elimination to get linear equation in *k*.

$k = -2$ ***A1***

[5 marks]

Examiners report

Many different attempts were seen, sometimes with success. Unfortunately many candidates wasted time with aimless substitutions showing little understanding of the problem.

(a) Write the vector equations of the following lines in parametric form.

$$r_1 = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} + m \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

$$r_2 = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + n \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$$

(b) Hence show that these two lines intersect and find the point of intersection, A.

(c) Find the Cartesian equation of the plane $\prod$ that contains these two lines.

(d) Let B be the point of intersection of the plane $\prod$ and the line $r = \begin{pmatrix} -8 \\ -3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 8 \\ 2 \end{pmatrix}$.

Find the coordinates of B.

(e) If C is the mid-point of AB, find the vector equation of the line perpendicular to the plane $\prod$ and passing through C.

Markscheme

(a) $x = 3 + 2m$

$y = 2 - m$

$z = 7 + 2m$ ***A1***

$x = 1 + 4n$

$y = 4 - n$

$z = 2 + n$ ***A1***

[2 marks]

(b) $3 + 2m = 1 + 4n \Rightarrow 2m - 4n = -2$ (i)

$2 - m = 4 - n \Rightarrow m - n = -2$ (ii) ***M1***

$7 + 2m = 2 + n \Rightarrow 2m - n = -5$ (iii)

(iii) − (ii) $\Rightarrow m = -3$ ***A1***

$\Rightarrow n = -1$ ***A1***

Substitute in (i), –6 + 4 = –2 . Hence lines intersect. ***R1***

Point of intersection A is (–3, 5,1) ***A1***

[5 marks]

(c) $\begin{vmatrix} i & j & k \\ 2 & -1 & 2 \\ 4 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix}$ ***M1A1***

$r \cdot \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix}$ ***(M1)***

$r \cdot \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix} = 29$

$x + 6y + 2z = 29$ ***A1***

Note: Award ***M1A0*** if answer is not in Cartesian form.

[4 marks]

(d) $x = -8 + 3\lambda$

$y = -3 + 8\lambda$ ***(M1)***

$z = 2\lambda$

Substitute in equation of plane.

$-8 + 3\lambda - 18 + 48\lambda + 4\lambda = 29$ ***M1***

$55\lambda = 55$

$\lambda = 1$ ***A1***

Coordinates of B are (–5, 5, 2) ***A1***

[4 marks]

(e) Coordinates of C are $\left(-4,\ 5,\ \frac{3}{2}\right)$ ***(A1)***

$$r = \begin{pmatrix} -4 \\ 5 \\ \frac{3}{2} \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix}$$ ***M1A1***

Note: Award ***M1A0*** unless candidate writes ***r*** = or $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

[3 marks]
Total [18 marks]

Examiners report

Most candidates found this question to their liking and many correct solutions were seen. In (b), some candidates solved two equations for *m* and *n* but then failed to show that these values satisfied the third equation. In (e), some candidates used an incorrect formula to determine the coordinates of the mid-point of *AB* .

A plane π has vector equation $\boldsymbol{r} = (-2\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}) + \lambda(2\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}) + \mu(6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k})$.

(a) Show that the Cartesian equation of the plane π is $3x + 2y - 6z = 12$.

(b) The plane π meets the x, y and z axes at A, B and C respectively. Find the coordinates of A, B and C.

(c) Find the volume of the pyramid OABC.

(d) Find the angle between the plane π and the x-axis.

(e) **Hence**, or otherwise, find the distance from the origin to the plane π.

(f) Using your answers from (c) and (e), find the area of the triangle ABC.

Markscheme

(a) **EITHER**

normal to plane given by

$$\begin{vmatrix} i & j & k \\ 2 & 3 & 2 \\ 6 & -3 & 2 \end{vmatrix}$$ ***M1A1***

$= 12\boldsymbol{i} + 8\boldsymbol{j} - 24\boldsymbol{k}$ ***A1***

equation of π is $3x + 2y - 6z = d$ ***(M1)***

as goes through $(-2, 3, -2)$ so $d = 12$ ***M1A1***

$\pi : 3x + 2y - 6z = 12$ ***AG***

OR

$x = -2 + 2\lambda + 6\mu$

$y = 3 + 3\lambda - 3\mu$

$z = -2 + 2\lambda + 2\mu$

eliminating μ

$x + 2y = 4 + 8\lambda$

$2y + 3z = 12\lambda$ ***M1A1A1***

eliminating λ

$3(x+2y)-2(2y+3z)=12$ ***M1A1A1***

$\pi: 3x+2y-6z=12$ ***AG***

[6 marks]

(b) therefore A(4, 0, 0), B(0, 6, 0) and C(0, 0, 2) ***A1A1A1***

Note: Award ***A1A1A0*** if position vectors given instead of coordinates.

[3 marks]

(c) area of base OAB $=\frac{1}{2}\times 4\times 6=12$ ***M1***

$V=\frac{1}{3}\times 12\times 2=8$ ***M1A1***

[3 marks]

(d) $\begin{pmatrix}3\\2\\-6\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=3=7\times 1\times\cos\phi$ ***M1A1***

$\phi=\arccos\frac{3}{7}$

so $\theta=90-\arccos\frac{3}{7}=25.4^\circ$ (accept 0.443 radians) ***M1A1***

[4 marks]

(e) $d=4\sin\theta=\frac{12}{7}\quad(=1.71)$ ***(M1)A1***

[2 marks]

(f) $8=\frac{1}{3}\times\frac{12}{7}\times\text{area}\Rightarrow\text{area}=14$ ***M1A1***

Note: If answer to part (f) is found in an earlier part, award ***M1A1***, regardless of the fact that it has not come from their answers to part (c) and part (e).

[2 marks]

Total [20 marks]

Examiners report

The question was generally well answered, although there were many students who failed to recognise that the volume was most logically found using a base as one of the coordinate planes.

The function *f* is defined on the domain [0, 2] by $f(x)=\ln(x+1)\sin(\pi x)$.

a. Obtain an expression for $f'(x)$. [3]

b. Sketch the graphs of f and f' on the same axes, showing clearly all x-intercepts. [4]

c. Find the x-coordinates of the two points of inflexion on the graph of f . [2]

d. Find the equation of the normal to the graph of f where $x = 0.75$, giving your answer in the form $y = mx + c$. [3]

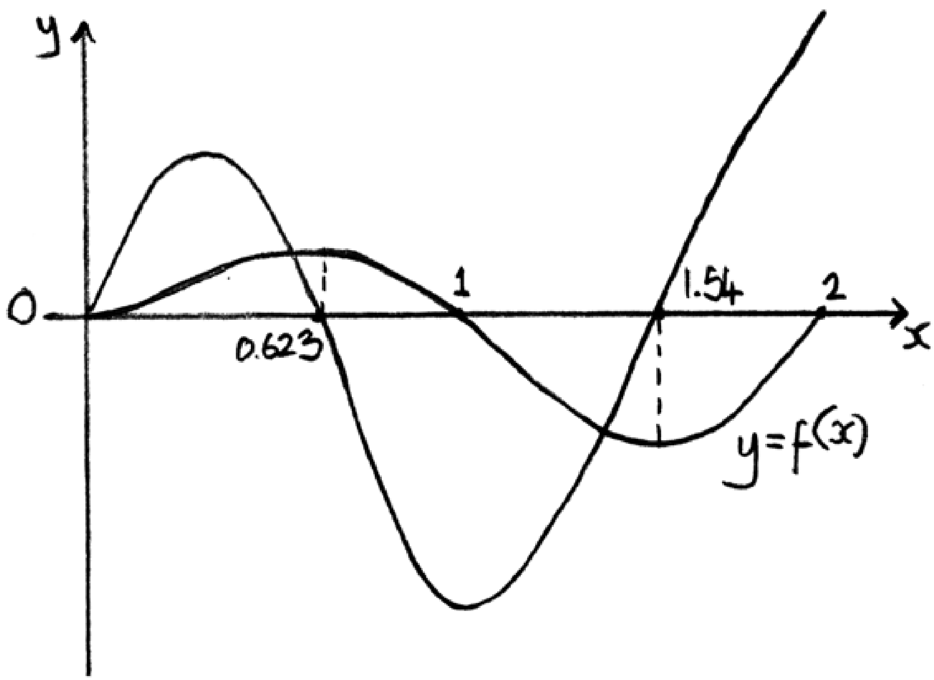
e. Consider the points A $(a\ ,\ f(a))$, B $(b\ ,\ f(b))$ and C $(c\ ,\ f(c))$ where a , b and c $(a < b < c)$ are the solutions of the equation $f(x) = f'(x)$. Find the area of the triangle ABC. [6]

Markscheme

a. $f'(x) = \frac{1}{x+1}\sin(\pi x) + \pi\ln(x+1)\cos(\pi x)$ ***M1A1A1***

[3 marks]

b.



A4

Note: Award ***A1A1*** for graphs, ***A1A1*** for intercepts.

[4 marks]

c. 0.310, 1.12 ***A1A1***

[2 marks]

d. $f'(0.75) = -0.839092$ ***A1***

so equation of normal is $y - 0.39570812 = \frac{1}{0.839092}(x - 0.75)$ ***M1***

$y = 1.19x - 0.498$ ***A1***

[3 marks]

e. $\text{A}(0,\ 0)$

$\text{B}(\overbrace{0.548\ldots}^{c}, \overbrace{0.432\ldots}^{d})$ ***A1***

$\text{C}(\overbrace{1.44\ldots}^{e}, \overbrace{-0.881\ldots}^{f})$ ***A1***

Note: Accept coordinates for B and C rounded to 3 significant figures.

area $\Delta\text{ABC} = \frac{1}{2}\left|(c\boldsymbol{i} + d\boldsymbol{j}) \times (e\boldsymbol{i} + f\boldsymbol{j})\right|$ ***M1A1***

$= \frac{1}{2}(de - cf)$ ***A1***

$= 0.554$ ***A1***

[6 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]
e. [N/A]

The position vector at time t of a point P is given by

$$\overrightarrow{\text{OP}} = (1 + t)\,\boldsymbol{i} + (2 - 2t)\,\boldsymbol{j} + (3t - 1)\,\boldsymbol{k},\ t \geqslant 0.$$

(a) Find the coordinates of P when $t = 0$.

(b) Show that P moves along the line L with Cartesian equations

$$x - 1 = \frac{y - 2}{-2} = \frac{z + 1}{3}$$

(c) (i) Find the value of t when P lies on the plane with equation $2x + y + z = 6$.

(ii) State the coordinates of P at this time.

(iii) Hence find the total distance travelled by P before it meets the plane.

The position vector at time t of another point, Q, is given by

$$\overrightarrow{\text{OQ}} = \begin{pmatrix} t^2 \\ 1 - t \\ 1 - t^2 \end{pmatrix},\ t \geqslant 0.$$

(d) (i) Find the value of t for which the distance from Q to the origin is minimum.

(ii) Find the coordinates of Q at this time.

(e) Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be the position vectors of Q at times $t = 0$, $t = 1$ and $t = 2$ respectively.

(i) Show that the equation $\boldsymbol{a} - \boldsymbol{b} = k\,(\boldsymbol{b} - \boldsymbol{c})$ has no solution for k .

(ii) Hence show that the path of Q is not a straight line.

Markscheme

(a) $\overrightarrow{\text{OP}} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ ***(M1)***

the coordinates of P are (1, 2, –1) ***A1***

[2 marks]

(b) **EITHER**

$x = 1 + t\,,\, y = 2 - 2t\,,\, z = 3t - 1$ ***M1***

$x - 1 = t\,,\, \frac{y-2}{-2} = t\,,\, \frac{z+1}{3} = t$ ***A1***

$x - 1 = \frac{y-2}{-2} = \frac{z+1}{3}$ ***AG*** ***N0***

OR

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$ ***M1A1***

$x - 1 = \frac{y-2}{-2} = \frac{z+1}{3}$ ***AG***

[2 marks]

(c) (i) $2\,(1 + t) + (2 - 2t) + (3t - 1) = 6 \Rightarrow t = 1$ ***M1A1*** ***N1***

(ii) coordinates are (2, 0, 2) ***A1***

Note: Award ***A0*** for position vector.

(iii) distance travelled is the distance between the two points ***(M1)***

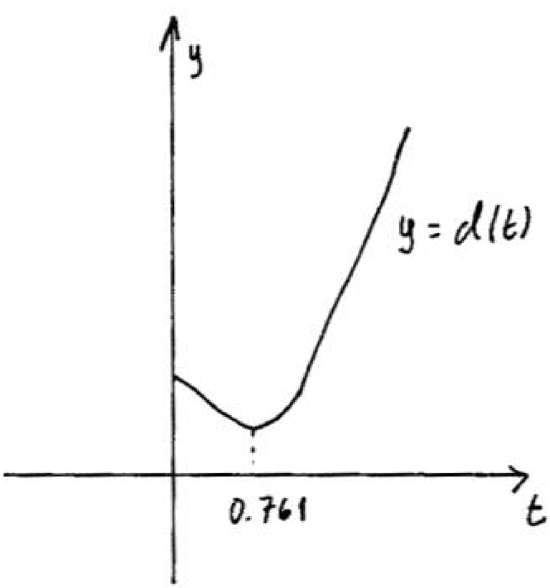
$\sqrt{(2-1)^2 + (0-2)^2 + (2+1)^2} = \sqrt{14}\ \ (= 3.74)$ ***(M1)A1***

[6 marks]

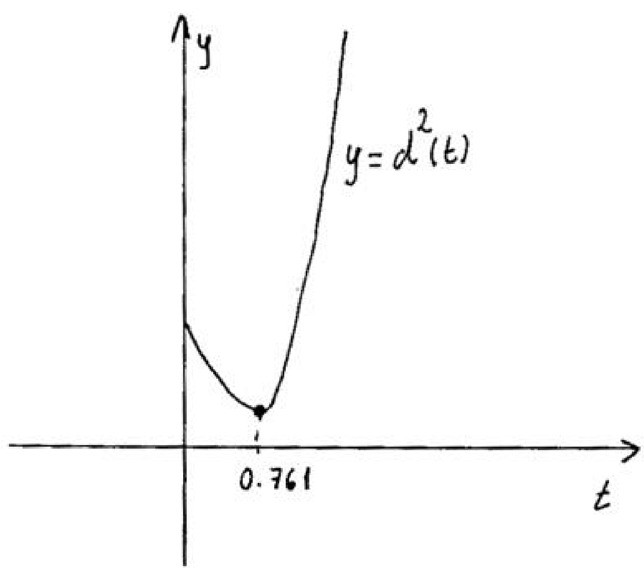
(d) (i) distance from Q to the origin is given by

$d(t) = \sqrt{t^4 + (1-t)^2 + (1-t^2)^2}$ (or equivalent) ***M1A1***

e.g. for labelled sketch of graph of d or d^2 ***(M1)(A1)***



or



the minimum value is obtained for $t = 0.761$ ***A1*** ***N3***

(ii) the coordinates are (0.579, 0.239, 0.421) ***A1***

Note: Accept answers given as a position vector.

[6 marks]

(e) (i) $\boldsymbol{a} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$ ***(M1)A1***

substituting in the equation $\boldsymbol{a} - \boldsymbol{b} = k\left(\boldsymbol{b} - \boldsymbol{c}\right)$, we have ***(M1)***

$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = k\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}\right) \Leftrightarrow \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = k\begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix}$ ***A1***

$\Rightarrow k = 1$ and $k = \frac{1}{3}$ which is impossible

so there is no solution for k ***R1***

(ii) $\mathrm{BA}$and $\overrightarrow{\mathrm{CB}}$ are not parallel ***R2***

(hence A, B, and C cannot be collinear)

Note: Only accept answers that follow from part (i).

[7 marks]

Total [23 marks]

Examiners report

Generally this question was answered well by those students who attempted it. It was common to see confusion between coordinates and position vectors. Part (d) was most easily answered with the use of a GDC, but fewer candidates took advantage of this. In part (e) many students had difficulties expressing their reasoning well to obtain the marks.